# Matrix Forensics

*Solving crimes of matrix math*

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github.com/r-barnes/MatrixForensics

# Contents

# 1 Introduction

**Goals:**

1. The primary goal of *Matrix Forensics* is to **solve crimes of matrix math**. That is, to make the sometimes mystifying manipulations of matrix math more understandable by cataloging useful identities, transformations, and facts.

2. **To be a community-accessible project.** Anyone can contribute to the project. The source code for the book is available on Github and the source code has been thoughtfully arranged with handy macros to help maintain an easy-to-use, aesthetic, and consistent notation and typography.

**Contributing:** Please contribute on Github at `https://github.com/r-barnes/MatrixForensics` either by opening an issue or making a pull request. If you are not comfortable with this, please send your contribution to `rijard.barnes@gmail.com`.

**Contributors:** Richard Barnes

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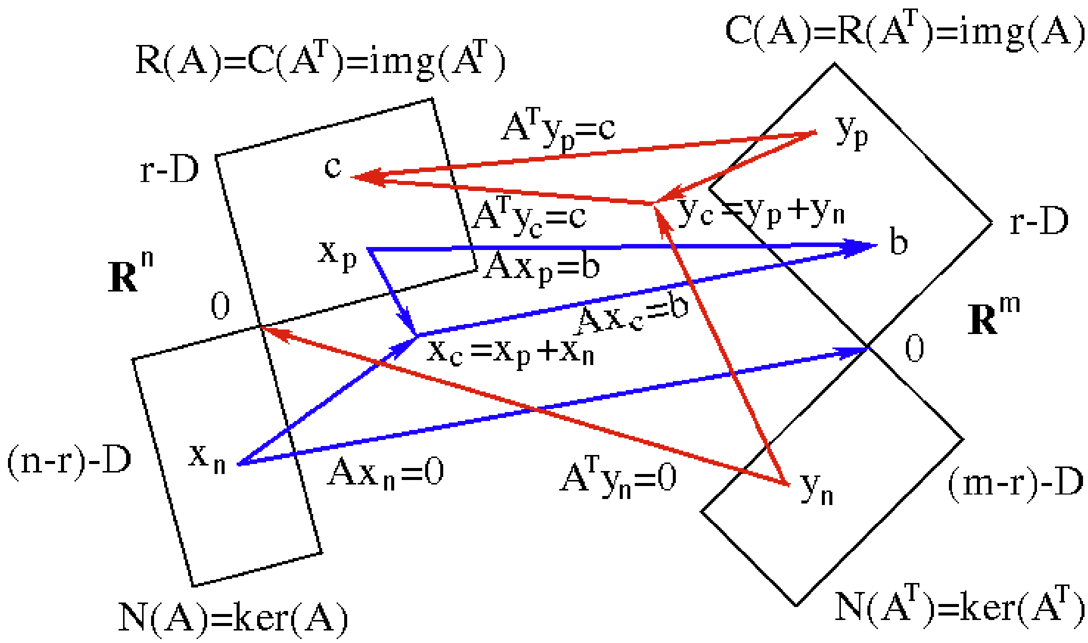
# 2 Nomenclature

| | |
|---|---|
| $\mathbf{A}$ | Matrix. |
| $\mathbf{a}$ | (Column) vector. |
| $a$ | Scalar. |
| $\lambda$ | An eigenvalue of a matrix. |
| | |
| $\mathbf{A}_{ij}$ | Matrix indexed. Returns $i$th row and $j$th column. |
| $\mathbf{A} \circ \mathbf{B}$ | Hadamard (element-wise) product of matrices A and B. |
| $\mathcal{N}(\mathbf{A})$ | Nullspace of the matrix $\mathbf{A}$. |
| $\mathcal{R}(\mathbf{A})$ | Range of the matrix $\mathbf{A}$. |
| $\det(\mathbf{A})$ | Determinant of the matrix $\mathbf{A}$. |
| $\mathrm{eig}(\mathbf{A})$ | Eigenvalues of the matrix $\mathbf{A}$. |
| $\mathbf{A}^H$ | Conjugate transpose of the matrix $\mathbf{A}$. |
| $\mathbf{A}^T$ | Transpose of the matrix $\mathbf{A}$. |
| $\mathbf{A}^+$ | Pseudoinverse of the matrix $\mathbf{A}$. |
| $\mathbf{x} \in \mathbb{R}^n$ | The entries of the $n$-vector $\mathbf{x}$ are all real numbers. |
| $\mathbf{A} \in \mathbb{R}^{m,n}$ | The entries of the matrix $\mathbf{A}$ with $m$ rows and $n$ columns are all real numbers. |
| $\mathbf{A} \in \mathbb{S}^n$ | The matrix $\mathbf{A}$ is symmetric and has $n$ rows and $n$ columns. |
| | |
| $\mathbf{I}_n$ | Identity matrix with $n$ rows and $n$ columns. |
| | |
| $\{0\}$ | The empty set |
| $\mathbb{R}$ | The real numbers |
| $\mathbb{C}$ | The complex numbers |

# 3 Basics

## 3.1 Fundamental Theorem of Linear Algebra



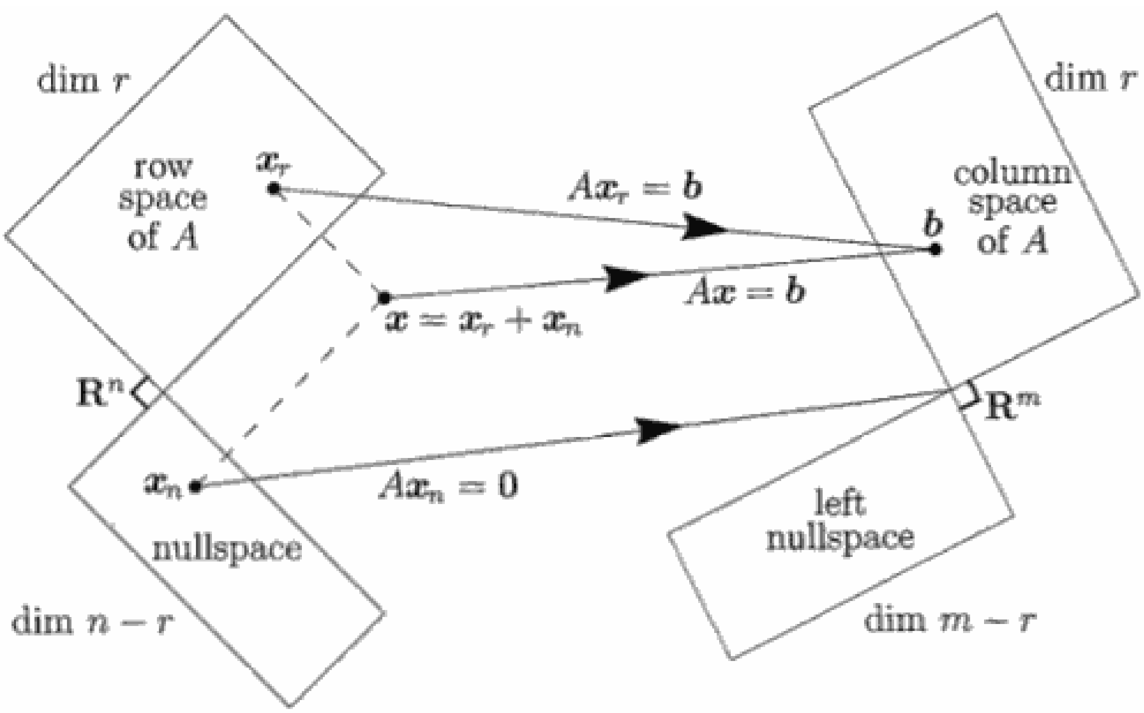
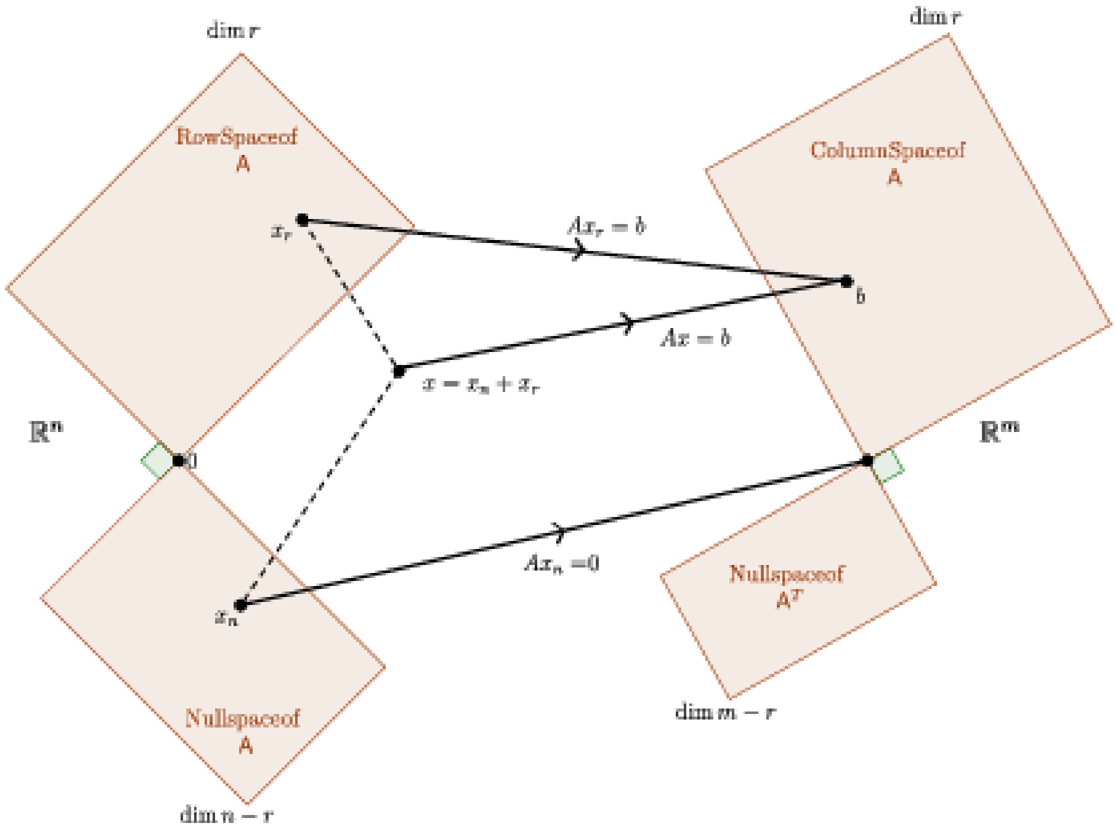
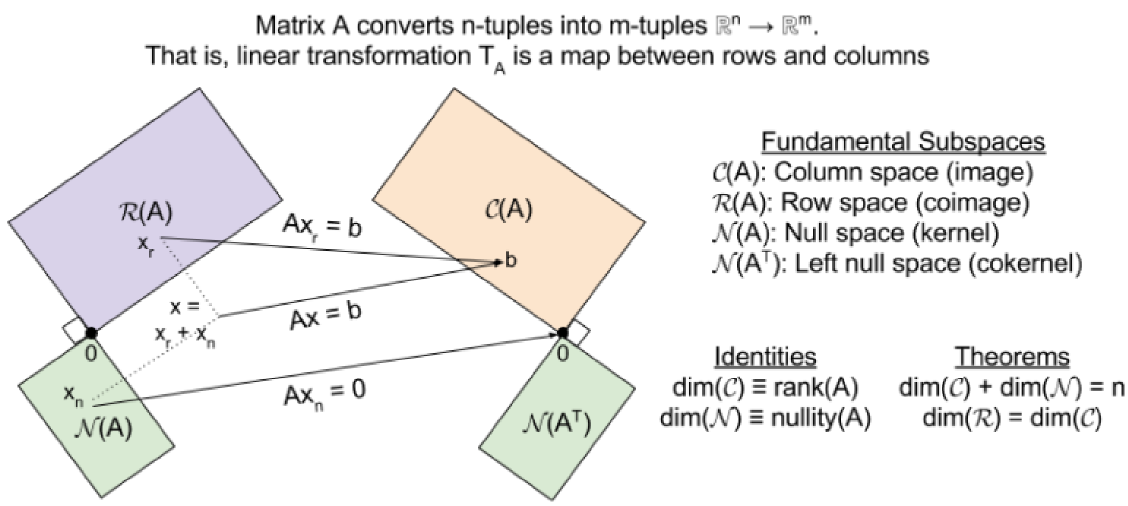




**Figure 3.4** The true action $Ax = A(x_{\text{row}} + x_{\text{null}})$ of any $m$ by $n$ matrix.


dim $r$
RowSpaceof
A
$x_r$
$Ax_r = b$
dim $r$
ColumnSpaceof
A
$b$
$Ax = b$
$x = x_n + x_r$
$\mathbb{R}^n$
0
$\mathbb{R}^m$
$Ax_n = 0$
Nullspaceof
$A^T$
$x_n$
dim $m - r$
Nullspaceof
A
dim $n - r$


Matrix A converts n-tuples into m-tuples $\mathbb{R}^n \rightarrow \mathbb{R}^m$.
That is, linear transformation $T_A$ is a map between rows and columns
$\mathcal{R}$(A)
$x_r$
$Ax_r = b$
$\mathcal{C}$(A)
b
x = $x_r + x_n$
Ax = b
0
$x_n$
$\mathcal{N}$(A)
$Ax_n = 0$
$\mathcal{N}(A^T)$
Fundamental Subspaces
$\mathcal{C}$(A): Column space (image)
$\mathcal{R}$(A): Row space (coimage)
$\mathcal{N}$(A): Null space (kernel)
$\mathcal{N}(A^T)$: Left null space (cokernel)
Identities
dim($\mathcal{C}$) ≡ rank(A)
dim($\mathcal{N}$) ≡ nullity(A)
Theorems
dim($\mathcal{C}$) + dim($\mathcal{N}$) = n
dim($\mathcal{R}$) = dim($\mathcal{C}$)

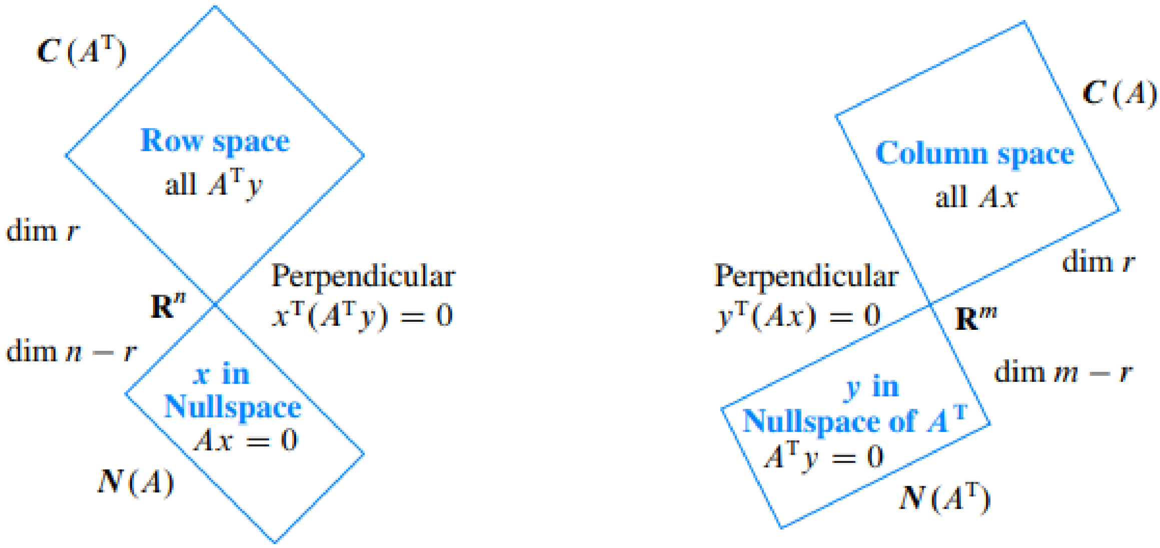



Figure 1: Dimensions and orthogonality for any $m$ by $n$ matrix $A$ of rank $r$.

## 3.2 Matrix Properties

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} \qquad \text{(left distributivity)} \tag{1}$$

$$(\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA} \qquad \text{(right distributivity)} \tag{2}$$

$$\mathbf{AB} \neq \mathbf{BA} \qquad \text{(in general)} \tag{3}$$

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC}) \qquad \text{(associativity)} \tag{4}$$

## 3.3 Rank

If $\mathbf{A} \in \mathbb{R}^{m,n}$ and $\mathbf{B} \in \mathbb{R}^{n,r}$, then

[1] $$\operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}) - n \leq \operatorname{rank}(\mathbf{AB}) \leq \min(\operatorname{rank}(\mathbf{A}), \operatorname{rank}(\mathbf{B})) \qquad \text{Sylvester's Inequality} \tag{5}$$

If $\mathbf{AB}$, $\mathbf{ABC}$, $\mathbf{BC}$ are defined, then

[1] $$\operatorname{rank}(\mathbf{AB}) + \operatorname{rank}(\mathbf{BC}) \leq \operatorname{rank}(\mathbf{B}) + \operatorname{rank}(\mathbf{ABC}) \qquad \text{Frobenius's inequality} \tag{6}$$

If $\dim(\mathbf{A}) = \dim(\mathbf{B})$, then

$$\operatorname{rank}(\mathbf{A}+\mathbf{B}) \leq \operatorname{rank}(\mathbf{A}) + \operatorname{rank}(\mathbf{B}) \qquad \text{Subadditivity} \tag{7}$$

If $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_l$ have $n_1, n_2, \ldots, n_l$ columns, so that $\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_l$ is well-defined, then

[1] $$\operatorname{rank}(\mathbf{A}_1\mathbf{A}_2 \ldots \mathbf{A}_l) \geq \sum_{i=1}^{l-1} \operatorname{rank}(\mathbf{A}_i\mathbf{A}_{i+1}) - \sum_{i=2}^{l-1} \operatorname{rank}(\mathbf{A}_i) \geq \sum_{i=1}^{l} \operatorname{rank}(\mathbf{A}_i) - \sum_{i=1}^{l-1} n_i \tag{8}$$

## 3.4 Identities

$$\left(\sum_{i=1}^{n} \mathbf{z}_i\right)^2 = \mathbf{z}^T \begin{bmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{bmatrix} \mathbf{z} \tag{9}$$

## 3.5 Matrix Multiplication

For $\mathbf{A} \in \mathbb{R}^{i,j}$ and $\mathbf{B} \in \mathbb{R}^{j,k}$ and $\mathbf{C} \in \mathbb{R}^{l,k}$

$$[\mathbf{AB}]_{ik} = \sum_j \mathbf{A}_{ij}\mathbf{B}_{jk} \tag{10}$$

$$[\mathbf{ABC}^T]_{il} = \sum_j \mathbf{A}_{ij}[\mathbf{BC}^T]_{jl} = \sum_j \mathbf{A}_{ij} \sum_k \mathbf{B}_{jk}\mathbf{C}_{lk} = \sum_j \sum_k \mathbf{A}_{ij}\mathbf{B}_{jk}\mathbf{C}_{lk} \tag{11}$$

## 3.6 Transpose Properties

$$(c\mathbf{A})^T = c\mathbf{A}^T \tag{12}$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T \tag{13}$$

$$(\mathbf{ABC}\dots)^T = \dots\mathbf{C}^T\mathbf{B}^T\mathbf{A}^T \tag{14}$$

$$(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T \tag{15}$$

$$(\mathbf{A}+\mathbf{B}+\dots)^T = \mathbf{A}^T + \mathbf{B}^T + \dots^T \tag{16}$$

$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \tag{17}$$

## 3.7 Conjugate Tranpose

$$(\mathbf{A}^H)^{-1} = (\mathbf{A}^{-1})^H \tag{18}$$

$$(\mathbf{A}+\mathbf{B})^H = \mathbf{A}^H + \mathbf{B}^H \tag{19}$$

$$(\mathbf{A}+\mathbf{B}+\dots)^H = \mathbf{A}^H + \mathbf{B}^H + \dots^H \tag{20}$$

$$(\mathbf{AB})^H = \mathbf{B}^H\mathbf{A}^H \tag{21}$$

$$(\mathbf{ABC}\dots)^H = \dots\mathbf{C}^H\mathbf{B}^H\mathbf{A}^H \tag{22}$$

## 3.8 Determinant Properties

The determinant is only defined for square matrices; here we assume that $\mathbf{A} \in \mathbb{R}^{n,n}$.

$$\det(\mathbf{I}_n) = 1 \tag{23}$$
$$\det(\mathbf{A}^T) = \det(\mathbf{A}) \tag{24}$$
$$\det(\mathbf{A}^H) = \det(\mathbf{A})^H \tag{25}$$
$$\det(\mathbf{A}^{-1}) = 1/\det(\mathbf{A}) \tag{26}$$
$$\det(\mathbf{AB}) = \det(\mathbf{BA}) \tag{27}$$
$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}) \qquad \mathbf{B} \in \mathbb{R}^{n,n} \tag{28}$$
$$\det(c\mathbf{A}) = c^n \det(\mathbf{A}) \tag{29}$$
$$\det(\mathbf{A}) = \prod \operatorname{eig}(\mathbf{A}) \tag{30}$$
$$\det(\mathbf{A}^n) = \det(\mathbf{A})^n \tag{31}$$
$$\det(-\mathbf{A}) = (-1)^n \det(\mathbf{A}) \tag{32}$$
$$\det(\mathbf{A}^c) = \det(\mathbf{A})^c \tag{33}$$
$$\det(\mathbf{I} + \mathbf{u}\mathbf{v}^T) = 1 + \mathbf{u}^T\mathbf{v} \tag{34}$$
$$\det(\mathbf{BAB}^{-1}) = \det(\mathbf{A}) \tag{35}$$
$$\det(\mathbf{BAB}^{-1} - c\mathbf{I}) = \det(\mathbf{A} - c\mathbf{I}) \tag{36}$$

For n=2:

$$\det(\mathbf{I} + \mathbf{A}) = 1 + \det(\mathbf{A}) + \operatorname{tr}(\mathbf{A}) \tag{37}$$
$$\det(\mathbf{A}) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{38}$$

For n=3:

$$\det(\mathbf{I} + \mathbf{A}) = 1 + \det(\mathbf{A}) + \operatorname{tr}(\mathbf{A}) + \frac{1}{2}\operatorname{tr}(\mathbf{A})^2 - \frac{1}{2}\operatorname{tr}(\mathbf{A}^2) \tag{39}$$
$$\det(\mathbf{A}) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \tag{40}$$

For n=4:

$$\det(\mathbf{I} + \mathbf{A}) = 1 + \det(\mathbf{A}) + \operatorname{tr}(\mathbf{A}) + \frac{1}{2}\operatorname{tr}(\mathbf{A})^2 - \frac{1}{2}\operatorname{tr}(\mathbf{A}^2) \tag{41}$$
$$+ \frac{1}{6}\operatorname{tr}(\mathbf{A})^3 - \frac{1}{2}\operatorname{tr}(\mathbf{A})\operatorname{tr}(\mathbf{A}^2) + \frac{1}{3}\operatorname{tr}(\mathbf{A}^3) \tag{42}$$

For small $\epsilon$:

$$\det(\mathbf{I} + \epsilon\mathbf{A}) \approx 1 + \det(\mathbf{A}) + \epsilon\operatorname{tr}(\mathbf{A}) + \frac{1}{2}\epsilon^2\operatorname{tr}(\mathbf{A})^2 - \frac{1}{2}\epsilon^2\operatorname{tr}(\mathbf{A}^2) \tag{43}$$
$$\det(\mathbf{I} + \epsilon\mathbf{A}) \approx 1 + \epsilon\operatorname{tr}(\mathbf{A}) + O(\epsilon^2) \tag{44}$$

Sylvester's determinant identity, for $\mathbf{A} \in \mathbb{R}^{m,n}, \mathbf{B} \in \mathbb{R}^{n,m}$

[2]
$$\det(\mathbf{I}_m + \mathbf{AB}) = \det(\mathbf{I}_n + \mathbf{BA}) \tag{45}$$
$$\det(\mathbf{X} + \mathbf{AB}) = \det(\mathbf{X})\det(\mathbf{I}_n + \mathbf{BX}^{-1}\mathbf{A}) \tag{46}$$

If $\mathbf{A}$ is triangular

$$\det(\mathbf{A}) = \prod_i \mathbf{A}_{i,i} = \prod_i \operatorname{diag}(\mathbf{A})_i \tag{47}$$

If all entries of $\mathbf{A} \in \mathbb{C}^{n,n}$ are in the unit disk

$$\det(\mathbf{A}) \leq n^{n/2} \tag{48}$$ [3]

Schur's determinant identities

$$\det(\mathbf{M}) = \det(\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}) = \det(\mathbf{A})\det(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}) \tag{49}$$

$$\det(\mathbf{M}) = \det(\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}) = \det(\mathbf{D})\det(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}) \tag{50}$$

$$\tag{51}$$

Geometrically, if a unit volume is acted on by $\mathbf{A}$, then $|\det(\mathbf{A})|$ indicates the volume after the transformation.

## 3.9 Trace Properties

The Trace is only defined for square matrices.

$$\mathrm{tr}(\mathbf{A}) = \sum_i \mathbf{A}_{ii} \tag{52}$$

$$\mathrm{tr}(\mathbf{A}) = \sum_i \mathrm{eig}(\mathbf{A}) \tag{53}$$

$$\mathrm{tr}(\mathbf{A} + \mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B}) \tag{54}$$

$$\mathrm{tr}(c\mathbf{A}) = c\,\mathrm{tr}(\mathbf{A}) \tag{55}$$

$$\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^T) \tag{56}$$

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA}) \tag{57}$$

$$\mathrm{tr}(\mathbf{A}^T\mathbf{B}) = \sum_{i,j} \mathbf{A}_{ij}\mathbf{B}_{ij} \tag{58}$$

$$\mathrm{tr}(\mathbf{A}^T\mathbf{B}) = \sum_{i,j} (\mathbf{A} \circ \mathbf{B})_{ij} \tag{59}$$

$$\mathbf{a}^T\mathbf{a} = \mathrm{tr}(\mathbf{a}\mathbf{a}^T) \tag{60}$$

For $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ of compatible dimensions,

$$\mathrm{tr}(\mathbf{A}^T\mathbf{B}) = \mathrm{tr}(\mathbf{A}\mathbf{B}^T) = \mathrm{tr}(\mathbf{B}^T\mathbf{A}) = \mathrm{tr}(\mathbf{B}\mathbf{A}^T) \tag{61}$$

$$\mathrm{tr}(\mathbf{ABCD}) = \mathrm{tr}(\mathbf{BCDA}) = \mathrm{tr}(\mathbf{CDAB}) = \mathrm{tr}(\mathbf{DABC}) \tag{62}$$

(Invariant under cyclic permutations)

## 3.10 Inverse Properties

The inverse of $\mathbf{A} \in \mathbb{C}^{n,n}$ is denoted $\mathbf{A}^{-1}$ and defined such that

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n \tag{63}$$

where $\mathbf{I}_n$ is the $n \times n$ identity matrix. $\mathbf{A}$ is nonsingular if $\mathbf{A}^{-1}$ exists; otherwise, $\mathbf{A}$ is singular.

If individual inverses exist

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{64}$$

more generally

$$(\mathbf{ABC}\ldots)^{-1} = \ldots\mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \tag{65}$$

$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1} \tag{66}$$
$$(\mathbf{A}^H)^{-1} = (\mathbf{A}^{-1})^H \tag{67}$$

Hua's Identity:

$$(\mathbf{A}+\mathbf{B})^{-1} = \mathbf{A}^{-1} - (\mathbf{A}+\mathbf{A}\mathbf{B}^{-1}\mathbf{A})^{-1} \tag{68}$$
$$(\mathbf{A}-\mathbf{B})^{-1} = \sum_{k=0}^{\infty}(\mathbf{A}^{-1}\mathbf{B})^k\mathbf{A}^{-1} \tag{69}$$
$$\tag{70}$$

## 3.11 Moore–Penrose PseudoInverse

For $\mathbf{A} \in \mathbb{R}^{m,n}$, the Moore–Penrose pseudoinverse $\mathbf{A}^+$ satisfies:

$$\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A} \tag{71}$$
$$\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+ \tag{72}$$
$$(\mathbf{A}\mathbf{A}^+)^T = \mathbf{A}\mathbf{A}^+ \text{ (symmetric)} \tag{73}$$
$$(\mathbf{A}^+\mathbf{A})^T = \mathbf{A}^+\mathbf{A} \text{ (symmetric)} \tag{74}$$

If $\mathbf{A}^+$ exists, it is unique. For complex matrices the symmetry condition is replaced by a requirement that the matrix be Hermitian.

If $\mathbf{A} \in \mathbb{C}^{m,n}$, then:

$$(\mathbf{A}^+)^+ = \mathbf{A} \tag{75}$$
$$(\mathbf{A}^T)^+ = (\mathbf{A}^+)^T \tag{76}$$
$$(\mathbf{A}^H)^+ = (\mathbf{A}^+)^H \tag{77}$$
$$(\mathbf{A}^*)^+ = (\mathbf{A}^+)^* \tag{78}$$
$$(\mathbf{A}^+\mathbf{A})\mathbf{A}^H = \mathbf{A}^H \tag{79}$$
$$(\mathbf{A}^+\mathbf{A})\mathbf{A}^T \neq \mathbf{A}^T \tag{80}$$
$$(c\mathbf{A})^+ = (1/c)\mathbf{A}^+ \tag{81}$$
$$\mathbf{A}^+ = (\mathbf{A}^T\mathbf{A})^+\mathbf{A}^T \tag{82}$$
$$\mathbf{A}^+ = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^+ \tag{83}$$
$$(\mathbf{A}^T\mathbf{A})^+ = \mathbf{A}^+(\mathbf{A}^T)^+ \tag{84}$$
$$(\mathbf{A}\mathbf{A}^T)^+ = (\mathbf{A}^T)^+\mathbf{A}^+ \tag{85}$$
$$\mathbf{A}^+ = (\mathbf{A}^H\mathbf{A})^+\mathbf{A}^H \tag{86}$$
$$\mathbf{A}^+ = \mathbf{A}^H(\mathbf{A}\mathbf{A}^H)^+ \tag{87}$$
$$(\mathbf{A}^H\mathbf{A})^+ = \mathbf{A}^+(\mathbf{A}^H)^+ \tag{88}$$
$$(\mathbf{A}\mathbf{A}^H)^+ = (\mathbf{A}^H)^+\mathbf{A}^+ \tag{89}$$
$$(\mathbf{A}\mathbf{B})^+ = (\mathbf{A}^+\mathbf{A}\mathbf{B})^+(\mathbf{A}\mathbf{B}\mathbf{B}^+)^+ \tag{90}$$

If **A** is full-rank, then:

$$(\mathbf{A}\mathbf{A}^+)(\mathbf{A}\mathbf{A}^+) = \mathbf{A}\mathbf{A}^+ \quad (91)$$

$$(\mathbf{A}^+\mathbf{A})(\mathbf{A}^+\mathbf{A}) = \mathbf{A}^+\mathbf{A} \quad (92)$$

$$\mathrm{tr}(\mathbf{A}\mathbf{A}^+) = \mathrm{rank}(\mathbf{A}\mathbf{A}^+) \quad (93)$$ [4]

$$\mathrm{tr}(\mathbf{A}^+\mathbf{A}) = \mathrm{rank}(\mathbf{A}^+\mathbf{A}) \quad (94)$$ [4]

### Special Properties

- $\mathbf{A}^+ = \mathbf{A}^{-1}$ if $\mathbf{A} \in \mathbb{R}^{n,n}$ and **A** is square and nonsingular.
- $\mathbf{A}^+ = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$, if $\mathbf{A} \in \mathbb{R}^{m,n}$ is full column rank ($r = n \leq m$). $\mathbf{A}^+$ is a left inverse of **A**, so $\mathbf{A}^+\mathbf{A} = \mathbf{V}_r\mathbf{V}_r^T = \mathbf{V}\mathbf{V}^T = \mathbf{I}_n$.
- $\mathbf{A}^+ = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}$, if $\mathbf{A} \in \mathbb{R}^{m,n}$ is full row rank ($r = m \leq n$). $\mathbf{A}^+$ is a right inverse of **A**, so $\mathbf{A}\mathbf{A}^+ = \mathbf{U}_r\mathbf{U}_r^T = \mathbf{U}\mathbf{U}^T = \mathbf{I}_m$.

## 3.12 Hadamard Identities

$$(\mathbf{A} \circ \mathbf{B})_{ij} = A_{ij}B_{ij} \;\forall\; i, j \quad (95)$$

$$\mathbf{A} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{A} \quad (96)$$ [5]

$$\mathbf{A} \circ (\mathbf{B} \circ \mathbf{C}) = (\mathbf{A} \circ \mathbf{B}) \circ \mathbf{C} \quad (97)$$

$$\mathbf{A} \circ (\mathbf{B} + \mathbf{C}) = \mathbf{A} \circ \mathbf{B} + \mathbf{A} \circ \mathbf{C} \quad (98)$$ [5]

$$a(\mathbf{A} \circ \mathbf{B}) = (a\mathbf{A}) \circ \mathbf{B} = \mathbf{A} \circ (a\mathbf{B}) \quad (99)$$ [5]

$$(\mathbf{A}^T \circ \mathbf{B}^T) = (\mathbf{A} \circ \mathbf{B})^T \quad (100)$$

$$(\mathbf{A}^T \circ \mathbf{B}^T) = (\mathbf{A} \circ \mathbf{B})^T \quad (101)$$

$$(\mathbf{x}^T\mathbf{A}\mathbf{x}) = \sum_{i,j} \left((\mathbf{x}\mathbf{x}^T) \circ \mathbf{A}\right) \quad (102)$$

$$\mathbf{x}^T(\mathbf{A} \circ \mathbf{B})\mathbf{y} = \mathrm{tr}((\mathrm{diag}(\mathbf{x})\mathbf{A})^T\mathbf{B}\,\mathrm{diag}(\mathbf{y})) \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^{m,n} \quad (103)$$ [6]

$$\mathrm{tr}(\mathbf{A}^T\mathbf{B}) = \mathbf{1}^T(\mathbf{A} \circ \mathbf{B})\mathbf{1} \quad (104)$$

$$= \sum_{i,j} \mathbf{A}_{ij}\mathbf{B}_{ij} \quad (105)$$

# 4 Derivatives

## 4.1 Useful Rules for Derivatives

For general $\mathbf{A}$ and $\mathbf{X}$ (no special structure):

$$\partial \mathbf{A} = 0 \quad \text{where } \mathbf{A} \text{ is a constant} \tag{106}$$
$$\partial (c\mathbf{X}) = c\partial \mathbf{X} \tag{107}$$
$$\partial (\mathbf{X} + \mathbf{Y}) = \partial \mathbf{X} + \partial \mathbf{Y} \tag{108}$$
$$\partial (\mathrm{tr}(\mathbf{X})) = \mathrm{tr}(\partial(\mathbf{X})) \tag{109}$$
$$\partial (\mathbf{X}\mathbf{Y}) = (\partial \mathbf{X})\mathbf{Y} + \mathbf{X}(\partial \mathbf{Y}) \tag{110}$$
$$\partial (\mathbf{X} \circ \mathbf{Y}) = (\partial \mathbf{X}) \circ \mathbf{Y} + \mathbf{X} \circ (\partial \mathbf{Y}) \tag{111}$$
$$\partial (\mathbf{X}^{-1}) = -\mathbf{X}^{-1}(\partial \mathbf{X})\mathbf{X}^{-1} \tag{112}$$
$$\partial (\det(\mathbf{X})) = \mathrm{tr}(\mathrm{adj}(\mathbf{X})\partial \mathbf{X}) \tag{113}$$
$$\partial (\det(\mathbf{X})) = \det(\mathbf{X})\, \mathrm{tr}(\mathbf{X}^{-1}\partial \mathbf{X}) \tag{114}$$
$$\partial (\ln(\det(\mathbf{X}))) = \mathrm{tr}(\mathbf{X}^{-1}\partial \mathbf{X}) \tag{115}$$
$$\partial (\mathbf{X}^{\mathsf{T}}) = (\partial \mathbf{X})^{\mathsf{T}} \tag{116}$$
$$\partial (\mathbf{X}^{H}) = (\partial \mathbf{X})^{H} \tag{117}$$

## 4.2 Gradient Notation

For a matrix $\mathbf{A} \in \mathbb{R}^{n,m}$, the gradient is defined as:

$$\nabla_{\mathbf{A}} f(\mathbf{A}) = \begin{bmatrix} \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{11}} & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{12}} & \cdots & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{1n}} \\ \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{21}} & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{22}} & \cdots & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{m1}} & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{m2}} & \cdots & \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{mn}} \end{bmatrix} \tag{118}$$

i.e.

$$(\nabla_{\mathbf{A}} f(\mathbf{A}))_{ij} = \frac{\partial f(\mathbf{A})}{\partial \mathbf{A}_{ij}} \tag{119}$$

Note that the size of the gradient is always the same size as the entity to which it is taken. Also note that the gradient of a function is only defined if the function is real-valued, that is, if it returns a scalar value.

## 4.3 Derivatives of Matrices and Vectors

### 4.3.1 First-Order

In the following, $\mathbf{J}$ is the Single-Entry Matrix (§ 5.17).

$$\frac{\partial \mathbf{x}^\mathsf{T}\mathbf{a}}{\partial \mathbf{x}} = \frac{\partial \mathbf{a}^\mathsf{T}\mathbf{x}}{\partial \mathbf{x}} = \mathbf{a} \tag{120}$$

$$\frac{\partial \mathbf{a}^\mathsf{T}\mathbf{X}\mathbf{b}}{\partial \mathbf{X}} = \mathbf{a}\mathbf{b}^\mathsf{T} \tag{121}$$

$$\frac{\partial \mathbf{a}^\mathsf{T}\mathbf{X}^\mathsf{T}\mathbf{b}}{\partial \mathbf{X}} = \mathbf{b}\mathbf{a}^\mathsf{T} \tag{122}$$

$$\frac{\partial \mathbf{a}^\mathsf{T}\mathbf{X}\mathbf{a}}{\partial \mathbf{X}} = \frac{\partial \mathbf{a}^\mathsf{T}\mathbf{X}^\mathsf{T}\mathbf{a}}{\partial \mathbf{X}} = \mathbf{a}\mathbf{a}^\mathsf{T} \tag{123}$$

$$\frac{\partial \mathbf{X}}{\partial \mathbf{X}_{ij}} = \mathbf{J}^{ij} \tag{124}$$

## 4.4 Derivatives of vector norms

$$\frac{\partial}{\partial \mathbf{x}}\|\mathbf{x}-\mathbf{a}\|_2 = \frac{\mathbf{x}-\mathbf{a}}{\|\mathbf{x}-\mathbf{a}\|_2} \tag{125}$$

$$\frac{\partial}{\partial \mathbf{x}}\frac{\mathbf{x}-\mathbf{a}}{\|\mathbf{x}-\mathbf{a}\|_2} = \frac{\mathbf{I}}{\|\mathbf{x}-\mathbf{a}\|_2} - \frac{(\mathbf{x}-\mathbf{a})(\mathbf{x}-\mathbf{a})^\mathsf{T}}{\|\mathbf{x}-\mathbf{a}\|_2^3} \tag{126}$$

$$\frac{\partial \|\mathbf{x}\|_2^2}{\partial \mathbf{x}} = \frac{\partial \left\|\mathbf{x}^\mathsf{T}\mathbf{x}\right\|_2}{\partial \mathbf{x}} = 2\mathbf{x} \tag{127}$$

## 4.5 Scalar by Vector

| Qualifier | Expression | Numerator layout | Denominator layout |
|---|---|---|---|
| | $\frac{\partial a}{\partial x}$ | $\mathbf{0}^\intercal$ | $\mathbf{0}$ |
| | $\frac{\partial a u(\mathbf{x})}{\partial \mathbf{x}}$ | $a\frac{\partial u}{\partial \mathbf{x}}$ | Same |
| | $\frac{\partial u(\mathbf{x})+v(\mathbf{x})}{\partial \mathbf{x}}$ | $\frac{\partial u}{\partial \mathbf{x}}+\frac{\partial v}{\partial \mathbf{x}}$ | Same |
| | $\frac{\partial u(\mathbf{x})v(\mathbf{x})}{\partial \mathbf{x}}$ | $u\frac{\partial v}{\partial \mathbf{x}}+v\frac{\partial u}{\partial \mathbf{x}}$ | Same |
| | $\frac{\partial g(u(\mathbf{x}))}{\partial \mathbf{x}}$ | $\frac{\partial g(u)}{\partial u}\frac{\partial u}{\partial \mathbf{x}}$ | Same |
| | $\frac{\partial f(g(u(\mathbf{x})))}{\partial \mathbf{x}}$ | $\frac{\partial f(g)}{\partial g}\frac{\partial g(u)}{\partial u}\frac{\partial u}{\partial \mathbf{x}}$ | Same |
| | $\frac{\partial \mathbf{u}(\mathbf{x})^\intercal \mathbf{v}(\mathbf{x})}{\partial \mathbf{x}}$ | $\mathbf{u}^\intercal\frac{\partial \mathbf{v}}{\partial \mathbf{x}}+\mathbf{v}^\intercal\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ | $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\mathbf{v}+\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\mathbf{u}$ |
| | $\frac{\partial \mathbf{u}(\mathbf{x})^\intercal \mathbf{A}\mathbf{v}(\mathbf{x})}{\partial \mathbf{x}}$ | $\mathbf{u}^\intercal\mathbf{A}\frac{\partial \mathbf{v}}{\partial \mathbf{x}}+\mathbf{v}^\intercal\mathbf{A}^\intercal\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ | $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\mathbf{A}\mathbf{v}+\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\mathbf{A}^\intercal\mathbf{u}$ |
| | $\frac{\partial^2 f}{\partial \mathbf{x}\partial \mathbf{x}^\intercal}$ | | $\mathbf{H}$, the Hessian matrix |
| | $\frac{\partial \mathbf{a}\cdot\mathbf{x}}{\partial \mathbf{x}}=\frac{\partial \mathbf{x}\cdot\mathbf{a}}{\partial \mathbf{x}}$ | $\mathbf{a}^\intercal$ | $\mathbf{a}$ |
| | $\frac{\partial \mathbf{b}^\intercal\mathbf{A}\mathbf{x}}{\partial \mathbf{x}}$ | $\mathbf{b}^\intercal\mathbf{A}$ | $\mathbf{A}^\intercal\mathbf{b}$ |
| | $\frac{\partial \mathbf{x}^\intercal\mathbf{A}\mathbf{x}}{\partial \mathbf{x}}$ | $\mathbf{x}^\intercal(\mathbf{A}+\mathbf{A}^\intercal)$ | $(\mathbf{A}+\mathbf{A}^\intercal)\mathbf{x}$ |
| $\mathbf{A}$ symmetric | $\frac{\partial \mathbf{x}^\intercal\mathbf{A}\mathbf{x}}{\partial \mathbf{x}}$ | $2\mathbf{x}^\intercal\mathbf{A}$ | $2\mathbf{A}\mathbf{x}$ |
| | $\frac{\partial \mathbf{x}^\intercal\mathbf{A}\mathbf{x}}{\partial \mathbf{x}}$ | $\mathbf{A}+\mathbf{A}^\intercal$ | Same |
| $\mathbf{A}$ symmetric | $\frac{\partial \mathbf{x}^\intercal\mathbf{A}\mathbf{x}}{\partial \mathbf{x}}$ | $\mathbf{A}$ | Same |
| | $\frac{\partial \mathbf{x}^\intercal\mathbf{x}}{\partial \mathbf{x}}$ | $2\mathbf{x}^\intercal$ | $2\mathbf{x}$ |
| | $\frac{\partial \mathbf{a}^\intercal\mathbf{u}(\mathbf{x})}{\partial \mathbf{x}}$ | $\mathbf{a}^\intercal\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ | $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\mathbf{a}$ |
| | $\frac{\partial \mathbf{a}^\intercal\mathbf{x}\mathbf{x}^\intercal\mathbf{b}}{\partial \mathbf{x}}$ | $\mathbf{x}^\intercal(\mathbf{a}\mathbf{b}^\intercal+\mathbf{b}\mathbf{a}^\intercal)$ | $(\mathbf{a}\mathbf{b}^\intercal+\mathbf{b}\mathbf{a}^\intercal)\mathbf{x}$ |
| | $\frac{\partial (\mathbf{A}\mathbf{x}+\mathbf{b})^\intercal\mathbf{C}(\mathbf{D}\mathbf{x}+\mathbf{e})}{\partial \mathbf{x}}$ | $(\mathbf{D}\mathbf{x}+\mathbf{e})^\intercal\mathbf{A}^\intercal\mathbf{A}+(\mathbf{A}\mathbf{x}+\mathbf{b})^\intercal\mathbf{C}\mathbf{D}$ | $\mathbf{A}^\intercal\mathbf{A}^\intercal(\mathbf{A}\mathbf{x}+\mathbf{b})+\mathbf{A}^\intercal\mathbf{C}(\mathbf{D}\mathbf{x}+\mathbf{e})$ |
| | $\frac{\partial \lVert\mathbf{x}-\mathbf{a}\rVert}{\partial \mathbf{x}}$ | $\frac{(\mathbf{x}-\mathbf{a})^\intercal}{\lVert\mathbf{x}-\mathbf{a}\rVert}$ | $\frac{\mathbf{x}-\mathbf{a}}{\lVert\mathbf{x}-\mathbf{a}\rVert}$ |

## 4.6 Vector by Vector

| Qualifier | Expression | Numerator layout | Denominator layout |
|---|---|---|---|
| | $\frac{\partial \mathbf{a}}{\partial \mathbf{x}}$ | $\mathbf{0}$ | Same |
| | $\frac{\partial \mathbf{x}}{\partial \mathbf{x}}$ | $\mathbf{I}$ | Same |
| | $\frac{\partial \mathbf{A}\mathbf{x}}{\partial \mathbf{x}}$ | $\mathbf{A}$ | $\mathbf{A}^\top$ |
| | $\frac{\partial \mathbf{x}^\top \mathbf{A}}{\partial \mathbf{x}}$ | $\mathbf{A}^\top$ | $\mathbf{A}$ |
| | $\frac{\partial a\mathbf{u}(\mathbf{x})}{\partial \mathbf{x}}$ | $a\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ | Same |
| | $\frac{\partial a(\mathbf{x})\mathbf{u}(\mathbf{x})}{\partial \mathbf{x}}$ | $a\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \mathbf{u}\frac{\partial a}{\partial \mathbf{x}}$ | $a\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial a}{\partial \mathbf{x}}\mathbf{u}^\top$ |
| | $\frac{\partial \mathbf{A}\mathbf{u}(\mathbf{x})}{\partial \mathbf{x}}$ | $\mathbf{A}\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ | $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\mathbf{A}^\top$ |
| | $\frac{\partial (\mathbf{u}(\mathbf{x})+\mathbf{v}(\mathbf{x}))}{\partial \mathbf{x}}$ | $\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \frac{\partial \mathbf{v}}{\partial \mathbf{x}}$ | Same |
| | $\frac{\partial \mathbf{g}(\mathbf{u}(\mathbf{x}))}{\partial \mathbf{x}}$ | $\frac{\partial \mathbf{g}(\mathbf{u})}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ | $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\frac{\partial \mathbf{g}(\mathbf{u})}{\partial \mathbf{u}}$ |
| | $\frac{\partial \mathbf{f}(\mathbf{g}(\mathbf{u}(\mathbf{x})))}{\partial \mathbf{x}}$ | $\frac{\partial \mathbf{f}(\mathbf{g})}{\partial \mathbf{g}(\mathbf{u})}\frac{\partial \mathbf{g}(\mathbf{u})}{\partial \mathbf{u}}\frac{\partial \mathbf{u}}{\partial \mathbf{x}}$ | $\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\frac{\partial \mathbf{g}(\mathbf{u})}{\partial \mathbf{u}}\frac{\partial \mathbf{f}(\mathbf{g})}{\partial \mathbf{g}}$ |

## 4.7 Matrix by Scalar

| Qualifier | Expression | Numerator layout |
|---|---|---|
| | $\frac{\partial a\mathbf{U}(x)}{\partial x}$ | $a\frac{\partial \mathbf{U}}{\partial x}$ |
| | $\frac{\partial \mathbf{A}\mathbf{U}(x)\mathbf{B}}{\partial x}$ | $\mathbf{A}\frac{\partial \mathbf{U}}{\partial x}\mathbf{B}$ |
| | $\frac{\partial (\mathbf{U}(x)+\mathbf{V}(x))}{\partial x}$ | $\frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathbf{V}}{\partial x}$ |
| | $\frac{\partial (\mathbf{U}(x)\mathbf{V}(x))}{\partial x}$ | $\mathbf{U}\frac{\partial \mathbf{V}}{\partial x} + \frac{\partial \mathbf{U}}{\partial x}\mathbf{V}$ |
| | $\frac{\partial (\mathbf{U}(x)\otimes\mathbf{V}(x))}{\partial x}$ | $\mathbf{U}\otimes\frac{\partial \mathbf{V}}{\partial x} + \frac{\partial \mathbf{U}}{\partial x}\otimes\mathbf{V}$ |
| | $\frac{\partial (\mathbf{U}(x)\circ\mathbf{V}(x))}{\partial x}$ | $\mathbf{U}\circ\frac{\partial \mathbf{V}}{\partial x} + \frac{\partial \mathbf{U}}{\partial x}\circ\mathbf{V}$ |
| | $\frac{\partial \mathbf{U}^{-1}(x)}{\partial x}$ | $-\mathbf{U}^{-1}\frac{\partial \mathbf{U}}{\partial x}\mathbf{U}^{-1}$ |
| | $\frac{\partial^2 \mathbf{U}^{-1}}{\partial x \partial y}$ | $\mathbf{U}^{-1}\left(\frac{\partial \mathbf{U}}{\partial x}\mathbf{U}^{-1}\frac{\partial \mathbf{U}}{\partial y} - \frac{\partial^2 \mathbf{U}}{\partial x \partial y} + \frac{\partial \mathbf{U}}{\partial y}\mathbf{U}^{-1}\frac{\partial \mathbf{U}}{\partial x}\right)\mathbf{U}^{-1}$ |
| | $\frac{\partial e^{x\mathbf{A}}}{\partial x}$ | $\mathbf{A}e^{x\mathbf{A}} = e^{x\mathbf{A}}\mathbf{A}$ |

## 4.8 Traces

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{X}) = \mathbf{I} \tag{128}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{X}\mathbf{A}) = \mathbf{A}^\intercal \tag{129}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{A}\mathbf{X}) = \mathbf{A}^\intercal \tag{130}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{A}\mathbf{X}\mathbf{B}) = \mathbf{A}^\intercal \mathbf{A}^\intercal \tag{131}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{A}\mathbf{X}^\intercal\mathbf{B}) = \mathbf{B}\mathbf{A} \tag{132}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{X}^\intercal\mathbf{A}) = \mathbf{A} \tag{133}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{A}\mathbf{X}^\intercal) = \mathbf{A} \tag{134}$$

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{A} \otimes \mathbf{X}) = \operatorname{tr}(\mathbf{A})\mathbf{I} \tag{135}$$

For traces with many instances of $\mathbf{X}$ we can apply an analogue of the product rule. For example:

$$\frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{A}\mathbf{X}\mathbf{B}\mathbf{X}\mathbf{A}^\intercal) = \frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{A}\mathbf{X}\mathbf{D}) + \frac{\partial}{\partial \mathbf{X}} \operatorname{tr}(\mathbf{E}\mathbf{X}\mathbf{A}^\intercal) = \mathbf{A}^\intercal\mathbf{A}^\intercal + \mathbf{A}^\intercal\mathbf{C} \tag{136}$$

where $\mathbf{D} = \mathbf{B}\mathbf{X}\mathbf{A}^\intercal$ and $\mathbf{E} = \mathbf{A}\mathbf{X}\mathbf{B}$.

## 4.9 Determinants

### 4.9.1 By Scalars

If $\mathbf{X}$ and $\mathbf{Y}$ are matrices with no special structure and $x$ is a scalar, then:

$$\frac{\partial \det(\mathbf{Y})}{\partial x} = \det(\mathbf{Y}) \operatorname{tr}\left(\mathbf{Y}^{-1}\frac{\partial \mathbf{Y}}{\partial x}\right) \tag{137}$$

$$\sum_k \frac{\partial \det(\mathbf{X})}{\partial \mathbf{X}_{ik}} \mathbf{X}_{jk} = \delta_{ij} \det(\mathbf{X}) \tag{138}$$

$$\frac{\partial^2 \mathbf{Y}}{\partial x^2} = \det(\mathbf{Y}) \left(\operatorname{tr}\left(\mathbf{Y}^{-1}\frac{\partial \frac{\partial \mathbf{Y}}{\partial x}}{\partial x}\right) + \operatorname{tr}\left(\mathbf{Y}^{-1}\frac{\partial \mathbf{Y}}{\partial x}\right) \operatorname{tr}\left(\mathbf{Y}^{-1}\frac{\partial \mathbf{Y}}{\partial x}\right) - \operatorname{tr}\left(\left(\mathbf{Y}^{-1}\frac{\partial \mathbf{Y}}{\partial x}\right)\left(\mathbf{Y}^{-1}\frac{\partial \mathbf{Y}}{\partial x}\right)\right)\right) \tag{139}$$

### 4.9.2 Linear forms

$$\frac{\partial \det(\mathbf{X})}{\partial \mathbf{X}} = \det(\mathbf{X})(\mathbf{X}^{-1})^\intercal \tag{140}$$

$$\sum_k \frac{\partial \det(\mathbf{X})}{\partial \mathbf{X}_{ik}} \mathbf{X}_{jk} = \delta_{ij} \det(\mathbf{X}) \tag{141}$$

$$\frac{\partial \det(\mathbf{A}\mathbf{X}\mathbf{B})}{\partial \mathbf{X}} = \det(\mathbf{A}\mathbf{X}\mathbf{B})(\mathbf{X}^{-1})^\intercal \tag{142}$$

$$= \det(\mathbf{A}\mathbf{X}\mathbf{B})(\mathbf{X}^\intercal)^{-1} \tag{143}$$

### 4.9.3 Square forms

If $\mathbf{X}$ is square and invertible:

$$\frac{\partial \det(\mathbf{X}^\intercal \mathbf{A}\mathbf{X})}{\partial \mathbf{X}} = 2\det(\mathbf{X}^\intercal \mathbf{A}\mathbf{X})\mathbf{X}^{-\intercal} \tag{144}$$

If $\mathbf{X}$ is not square and $\mathbf{A}$ is symmetric, then

$$\frac{\partial \det(\mathbf{X}^\intercal \mathbf{A}\mathbf{X})}{\partial \mathbf{X}} = 2\det(\mathbf{X}^\intercal \mathbf{A}\mathbf{X})\mathbf{A}\mathbf{X}(\mathbf{X}^\intercal \mathbf{A}\mathbf{X})^{-1} \tag{145}$$

If $\mathbf{X}$ is not square and $\mathbf{A}$ is not symmetric, then

$$\frac{\partial \det(\mathbf{X}^\intercal \mathbf{A}\mathbf{X})}{\partial \mathbf{X}} = \det(\mathbf{X}^\intercal \mathbf{A}\mathbf{X})\left(\mathbf{A}\mathbf{X}(\mathbf{X}^\intercal \mathbf{A}\mathbf{X})^{-1} + \mathbf{A}^\intercal \mathbf{X}(\mathbf{X}^\intercal \mathbf{A}^\intercal \mathbf{X})^{-1}\right) \tag{146}$$

### 4.9.4 Nonlinear Forms

$$\frac{\partial \ln\det(\mathbf{X}^\intercal \mathbf{X})}{\partial \mathbf{X}} = 2(\mathbf{X}^+)^\intercal \tag{147}$$

$$\frac{\partial \ln\det(\mathbf{X}^\intercal \mathbf{X})}{\partial \mathbf{X}^+} = -2\mathbf{X}^\intercal \tag{148}$$

$$\frac{\partial \ln|\det(\mathbf{X})|}{\partial \mathbf{X}} = \mathbf{X}^{-\intercal} \tag{149}$$

$$\frac{\partial \det(\mathbf{X}^k)}{\partial \mathbf{X}} = k\det(\mathbf{X}^k)\mathbf{X}^{-\intercal} \tag{150}$$

# 5 Matrix Rogue Gallery

## 5.1 Non-Singular vs. Singular Matrices

For $\mathbf{A} \in \mathbb{R}^{n,n}$ (initially drawn from [7, p. 574]):

| **Non-Singular** | **Singular** |
|---|---|
| $\mathbf{A}$ is invertible | $\mathbf{A}$ is not invertible |
| The columns are independent | The columns are dependent |
| The rows are independent | The rows are dependent |
| $\det(\mathbf{A}) \neq 0$ | $\det(\mathbf{A}) = 0$ |
| $\mathbf{Ax} = 0$ has one solution: $\mathbf{x} = 0$ | $\mathbf{Ax} = 0$ has infinitely many solutions |
| $\mathbf{Ax} = \mathbf{b}$ has one solution: $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ | $\mathbf{Ax} = \mathbf{b}$ has either no or infinitely many solutions |
| $\mathbf{A}$ has $n$ nonzero pivots | $\mathbf{A}$ has $r < n$ pivots |
| $\mathbf{A}$ has full rank $r = n$ | $\mathbf{A}$ has rank $r < n$ |
| The reduced row echelon form is $\mathbf{R} = \mathbf{I}$ | $\mathbf{R}$ has at least one zero row |
| The column space is all of $\mathbb{R}^n$ | The column space has dimension $r < n$ |
| The row space is all of $\mathbb{R}^n$ | The row space has dimension $r < n$ |
| All eigenvalues are nonzero | Zero is an eigenvalue of $\mathbf{A}$ |
| $\mathbf{A}^T\mathbf{A}$ is symmetric positive definite | $\mathbf{A}^T\mathbf{A}$ is only semidefinite |
| $\mathbf{A}$ has $n$ positive singular values | $\mathbf{A}$ has $r < n$ singular values |

## 5.2 2x2 Matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{151}$$

$$\det(\mathbf{A}) = \mathbf{A}_{1,1}\mathbf{A}_{2,2} - \mathbf{A}_{1,2}\mathbf{A}_{2,1} \tag{152}$$

$$\operatorname{tr}(\mathbf{A}) = \mathbf{A}_{1,1} + \mathbf{A}_{2,2} \tag{153}$$

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} \mathbf{A}_{2,2} & -\mathbf{A}_{1,2} \\ -\mathbf{A}_{2,1} & \mathbf{A}_{1,1} \end{bmatrix} \tag{154}$$

### 5.2.1 Eigenvalues

$$\lambda = \frac{\operatorname{tr}(\mathbf{A}) \pm \sqrt{\operatorname{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})}}{2} \tag{155}$$

$$0 = \lambda^2 - \lambda \operatorname{tr}(\mathbf{A}) + \det(\mathbf{A}) \tag{156}$$

$$\operatorname{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 \tag{157}$$

$$\det(\mathbf{A}) = \lambda_1 \lambda_2 \tag{158}$$

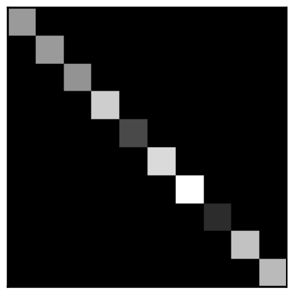
### 5.2.2 Eigenvectors

$$\mathbf{v}_1 \propto \begin{bmatrix} \mathbf{A}_{12} \\ \lambda_1 - \mathbf{A}_{11} \end{bmatrix} \tag{159}$$

$$\mathbf{v}_2 \propto \begin{bmatrix} \mathbf{A}_{12} \\ \lambda_2 - \mathbf{A}_{11} \end{bmatrix} \tag{160}$$

$$\tag{161}$$

## 5.3 Diagonal Matrix



$$A = \operatorname{diag}(a_1, \ldots, a_n) = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{bmatrix} \tag{162}$$

Square matrix. Entries above diagonal are equal to entries below diagonal.

Number of "free entries": $\frac{n(n+1)}{2}$.

### Special Properties

$$\operatorname{eig}(A) = a_1, \ldots, a_n \tag{163}$$

$$\det(A) = \prod_i a_i \tag{164}$$

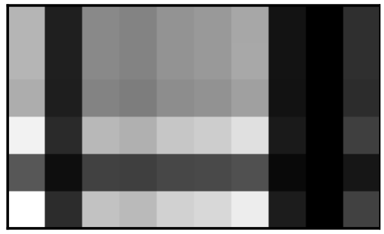
$$A^{-1} = \begin{bmatrix} \frac{1}{a_1} & & \\ & \ddots & \\ & & \frac{1}{a_n} \end{bmatrix} \tag{165}$$

$$\mathbf{x}^T \mathbf{A} \mathbf{x} = \sum_i a_i \mathbf{x}_i^2 \tag{166}$$

## 5.4 Doubly stochastic matrix

A square matrix of nonnegative real numbers whose rows and columns each sum to 1.

## 5.5 Dyads



$\mathbf{A} \in \mathbb{R}^{m,n}$ is a dyad if it can be written as

$$\mathbf{A} = \mathbf{u}\mathbf{v}^T \quad \mathbf{u} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^n \tag{167}$$

### Special Properties

- The columns of $\mathbf{A}$ are copies of $\mathbf{u}$ scaled by the values of $\mathbf{v}$.
- The rows of $\mathbf{A}$ are copies of $\mathbf{u}^T$ scaled by the values of $\mathbf{v}$.
- If $\mathbf{A}$ is a dyad, it acts on a vector $\mathbf{x}$ as $\mathbf{A}\mathbf{x} = (\mathbf{u}\mathbf{v}^T)\mathbf{x} = (\mathbf{v}^T\mathbf{u})\mathbf{x}$.
- $\mathbf{A}\mathbf{x} = c\mathbf{u}$ ($\mathbf{A}$ scales $\mathbf{x}$ and points it along $\mathbf{u}$).
- $\mathbf{A}_{ij} = \mathbf{u}_i\mathbf{v}_j$.
- If $\mathbf{u}, \mathbf{v} \neq 0$, then $\text{rank}(\mathbf{A}) = 1$.
- If $m = n$, $\mathbf{A}$ has one eigenvalue $\lambda = \mathbf{v}^T\mathbf{u}$ and eigenvector $\mathbf{u}$.
- A dyad can always be written in a normalized form $c\tilde{\mathbf{u}}\tilde{\mathbf{v}}^T$.

## 5.6 Hermitian Matrix

$\mathbf{H} \in \mathbb{C}^{m,n}$ is Hermitian iff

$$\mathbf{H} = \mathbf{H}^H \tag{168}$$

where $\mathbf{H}^H$ is the conjugate transpose of $\mathbf{H}$.

For $\mathbf{H} \in \mathbb{R}^{m,n}$, Hermitian and symmetric matrices are equivalent.

### Special Properties

$$\mathbf{H}_{ii} \in \mathbb{R} \tag{169}$$
$$\mathbf{H}\mathbf{H}^H = \mathbf{H}^H\mathbf{H} \tag{170}$$
$$\mathbf{x}^H\mathbf{H}\mathbf{x} \in \mathbb{R} \quad \forall \mathbf{x} \in \mathbb{C} \tag{171}$$
$$\mathbf{H}_1 + \mathbf{H}_2 = \text{Hermitian} \tag{172}$$
$$\mathbf{H}^{-1} = \text{Hermitian} \tag{173}$$
$$\mathbf{A} + \mathbf{A}^H = \text{Hermitian} \tag{174}$$
$$\mathbf{A} - \mathbf{A}^H = \text{Skew-Hermitian} \tag{175}$$
$$\mathbf{A}\mathbf{B} = \text{Hermitian iff } \mathbf{A}\mathbf{B} = \mathbf{B}\mathbf{A} \tag{176}$$
$$\det(\mathbf{H}) \in \mathbb{R} \tag{177}$$
$$\text{eig}(\mathbf{H}) \in \mathbb{R} \tag{178}$$

## 5.7 Hurwitz matrix

TODO

## 5.8 Idempotent Matrix

A matrix $\mathbf{A}$ is idempotent iff

$$\mathbf{AA} = \mathbf{A} \tag{179}$$

### Special Properties

$$\mathbf{A}^n = \mathbf{A} \quad \forall n \geq 1 \tag{180}$$
$$\mathbf{I} - \mathbf{A} \text{ is idempotent} \tag{181}$$
$$\mathbf{A}^H \text{ is idempotent} \tag{182}$$
$$\mathbf{I} - \mathbf{A}^H \text{ is idempotent} \tag{183}$$
$$\text{rank}(\mathbf{A}) = \text{tr}(\mathbf{A}) \tag{184}$$
$$\mathbf{A}(\mathbf{I} - \mathbf{A}) = 0 \tag{185}$$
$$\mathbf{A}^+ = \mathbf{A} \tag{186}$$
$$f(s\mathbf{I} + t\mathbf{A}) = (\mathbf{I} - \mathbf{A})f(s) + \mathbf{A}f(s+t) \tag{187}$$
$$\mathbf{AB} = \mathbf{BA} \implies \mathbf{AB} \text{ is idempotent} \tag{188}$$
$$\text{eig}(\mathbf{A})_i \in \{0, 1\} \tag{189}$$
$$\mathbf{A} \text{ is always diagonalizable} \tag{190}$$

$\mathbf{A} - \mathbf{I}$ may not be idempotent.

See also: nilpotent (§ 5.11), unipotent (§ 5.25).

## 5.9 Laplacian Matrix of a Graph

Let $\mathbf{L}$ be the Laplacian matrix of a graph $G$ with neither multiple edges nor loops defined by $(V, E, w)$ where $V$ is the set of vertices, $E$ the set of edges, and $w$ is a weight function. Is is also the case that $L = D - A$ where $D$ is the degree matrix and $A$ is the adjaceny matrix. In the case of directed graphs either the indegree or outdegree might be used.

The elements of $\mathbf{L}$ are given by

$$\mathbf{L}_{i,j} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases} \tag{191}$$

If $G$ is weighted, the elements of its Laplacian $\mathbf{L}$ are given by

$$\mathbf{L}_{i,j} = \begin{cases} \sum_{j, j \neq i} w(i,j) & \text{if } i = j \\ -w(i,j) & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \text{ with weight } w(i,j) \\ 0 & \text{otherwise} \end{cases} \tag{192}$$

For an undirected graph $G$ and its Laplacian $\mathbf{L}$:

- **L** is symmetric
- $L \succeq 0$
- The row sum and column sums of **L** are both zero.
- **L** is singular
- The number of connected components in $G$ is the dimension of $\mathcal{N}(L)$ and the algebraic multiplicity of the 0 eigenvalue.
- The smallest non-zero eigenvalue of **L** is called the spectral gap.
- The second smallest eigenvalue of **L** (could be zero) is the algebraic connectivity (Fiedler value) of $G$ and approximates the sparest cut of $G$.
- For a graph with multiple connected components, **L** is a block diagonal matrix.
- Using preconditioners, the linear equaitons of any Laplacian matrix $\mathbf{L} \in \mathbb{R}^{n,n}$ can be solved to accuracy $\epsilon$ in time $O((\text{nnz}(\mathbf{L})+n\log n(\log\log n)^2)\log\epsilon^{-1})$. The best balance between preconditioners and solving linear equations yields an algorithm of complexity $O(\text{nnz}(\mathbf{L})\log^c n \log\epsilon^{-1})$. [8]

$$\mathbf{x}^T\mathbf{L}\mathbf{x} = \sum_{(u,v)\in E} w(u,v)\left(\mathbf{x}(u)-\mathbf{x}(v)\right)^2 \qquad \mathbf{x}\in\mathbb{R}^V \tag{193}$$

Equation 193 provides a measure of the “smoothness” of $\mathbf{x}$ over the edges of $G$. The more $\mathbf{x}$ jumps over an edge, the larger the quadratic form becomes.

## 5.10 Metzler matrix

TODO

## 5.11 Nilpotent

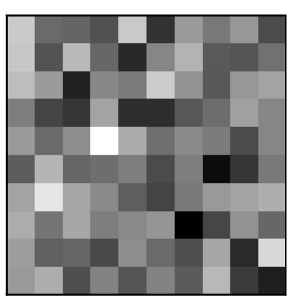
A matrix **A** is nilpotent iff

$$\mathbf{A}^2 = 0 \tag{194}$$

### Special Properties

$$f(s\mathbf{I}+t\mathbf{A}) = \mathbf{I}f(s) + t\mathbf{A}f'(s) \tag{195}$$

## 5.12 Orthogonal Matrix



(Not much visible structure)

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{196}$$

A square matrix $\mathbf{U} \in \mathbb{R}^{n,n}$ is orthogonal iff:

$$\mathbf{U}^T\mathbf{U} = \mathbf{U}\mathbf{U}^T = \mathbf{I} \tag{197}$$

The columns form an orthonormal basis of $\mathbb{R}^n$.

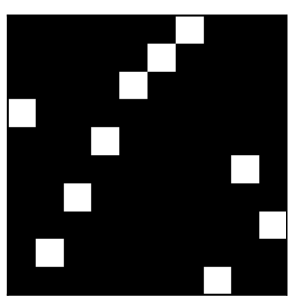
### Special Properties

- The eigenvalues of $\mathbf{U}$ are placed on the unit circle.
- The eigenvectors of $\mathbf{U}$ are unitary (have length one).
- $\mathbf{U}^{-1}$ is orthogonal.
- The product of two orthogonal matrices is itself orthogonal.

$$\mathbf{U}^T = \mathbf{U}^{-1} \tag{198}$$
$$\mathbf{U}^{-T} = \mathbf{U} \tag{199}$$
$$\mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{200}$$
$$\mathbf{U}\mathbf{U}^T = \mathbf{I} \tag{201}$$
$$\det(\mathbf{U}) = \pm 1 \tag{202}$$
$$\|\mathbf{U}\mathbf{x}\|_2^2 = (\mathbf{U}\mathbf{x})^T(\mathbf{U}\mathbf{x}) = \mathbf{x}^T\mathbf{U}^T\mathbf{U}\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|_2^2 \quad \forall\mathbf{x} \tag{203}$$
$$\|\mathbf{U}\mathbf{A}\mathbf{V}\|_F = \|\mathbf{A}\|_F \quad \forall\mathbf{A}, \mathbf{U}, \mathbf{V} \text{ with } \mathbf{U}, \mathbf{V}\text{orthogonal} \tag{204}$$

Orthogonal matrices preserve the lengths and angles of the vectors they operator on. The converse is true: any matrix which preserves lengths and angles is orthogonal.

## 5.13 Permutation Matrix



TODO

## 5.14 Positive Definite

$\mathbf{P} \in \mathbb{S}^n$ is positive definite (denoted $\mathbf{P} \succ 0$) if any of the following are true:

- $\mathbf{x}^T\mathbf{P}\mathbf{x} > 0, \forall \mathbf{x} \in \mathbb{R}^n$.
- $\mathrm{eig}(\mathbf{P}) > 0$
- There exists a unique matrix $\mathbf{U} \in \mathbb{R}^{n,n}$, such that $\mathbf{A} = \mathbf{U}\mathbf{U}^T$ (Cholesky Decomposition).

### Special Properties

- $\mathbf{P}^{-1} \succ 0$
- $c\mathbf{P} \succ 0$
- $\mathbf{A}_{ii} \in \mathbb{R}$
- $\mathbf{A}_{ii} > 0$
- $\mathrm{tr}(\mathbf{P}) \geq 0$.
- $\det(\mathbf{P}) > 0$
- The eigenvalues of $\mathbf{P}^{-1}$ are the inverses of the eigenvalues of $\mathbf{P}$.
- For $\mathbf{P} \in \mathbb{R}^{m,n}$, $\mathbf{P}^T\mathbf{P} \succ 0 \iff \mathbf{P}$ is full-column rank ($\mathrm{rank}(\mathbf{P}) = n$)
- For $\mathbf{P} \in \mathbb{R}^{m,n}$, $\mathbf{P}\mathbf{P}^T \succ 0 \iff \mathbf{P}$ is full-row rank ($\mathrm{rank}(\mathbf{P}) = m$)

#### Ellipsoids

$\mathbf{P} \succ 0$ defines a full-dimensional, bounded ellipsoid defined by the set

$$\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x} - \mathbf{z})^T\mathbf{P}^{-1}(\mathbf{x} - \mathbf{z}) \leq \beta\} \tag{205}$$

The eigenvectors of $\mathbf{P}$ define the directions of the semi-axes of the ellipsoid; the lengths of these axes are given by $\sqrt{\beta\lambda_i}$ where $\lambda_i$ are the eigenvalues of $\mathbf{P}$. The ellipsoid is centered at $\mathbf{z}$. Since $\mathbf{P} \succ 0 \implies \mathbf{P}^{-1} \succ 0$, the Cholesky decomposition says that $\mathbf{P}^{-1} = \mathbf{A}^T\mathbf{A}$; therefore, an equivalent definition of the ellipsoid is $\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{A}\mathbf{x}\|_2 \leq 1\}$.

## 5.15 Positive Semi-Definite

$\mathbf{A}$ is positive semi-definite (denoted $\mathbf{A} \succeq 0$) if any of the following are true:

- $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0, \forall \mathbf{x} \in \mathbb{R}^n$.
- $\mathrm{eig}(\mathbf{A}) \geq 0$
- There exists a non-unique matrix $\mathbf{U} \in \mathbb{R}^{n,n}$, such that $\mathbf{A} = \mathbf{U}\mathbf{U}^T$ (Cholesky Decomposition).

### Special Properties

- For $\mathbf{A} \in \mathbb{R}^{m,n}$, $\mathbf{A}^T\mathbf{A} \succeq 0$
- For $\mathbf{A} \in \mathbb{R}^{m,n}$, $\mathbf{A}\mathbf{A}^T \succeq 0$
- $\mathrm{diag}(\mathbf{A})_i \geq 0$
- $\sum_{ij} \mathbf{A}_{ij} \geq 0$

- $\mathrm{tr}(\mathbf{A}) \geq 0$
- For $\mathbf{A}, \mathbf{B} \succeq 0$, $\mathrm{tr}(\mathbf{AB}) \geq 0$
- For $\mathbf{A}, \mathbf{B} \succeq 0$, $\mathrm{tr}(\mathbf{AB}) = 0 \iff \mathbf{AB} = 0$
- The positive semi-definite matrices $\mathbb{S}_+^n$ form a convex cone. For any two PSD matrices $\mathbf{A}, \mathbf{B} \in \mathbb{S}_+^n$ and some $\alpha \in [0, 1]$:

$$\mathbf{x}^T(\alpha\mathbf{A} + (1-\alpha)\mathbf{B})\mathbf{x} = \alpha\mathbf{x}^T\mathbf{A}\mathbf{x} + (1-\alpha)\mathbf{x}^T\mathbf{B}\mathbf{x} \geq 0 \quad \forall\mathbf{x} \tag{206}$$

$$\alpha\mathbf{A} + (1-\alpha)\mathbf{B} \in \mathbb{S}_+^n \tag{207}$$

- For $\mathbf{A} \in \mathbb{S}_+^n$ and $\alpha \geq 0$, $\alpha\mathbf{A} \succeq 0$, so $\mathbb{S}_+^n$ is a cone.
- $\mathbf{A} \succeq 0$ if and only if there is a PSD matrix $\mathbf{S}^{1/2}$ such that $\mathbf{S}^{1/2}\mathbf{S}^{1/2} = \mathbf{A}$. This $\mathbf{S}$ is unique.

### 5.15.1 Loewner order

If $\mathbf{A} - \mathbf{B} \succeq 0$, then we say $\mathbf{A} \succeq \mathbf{B}$. A sufficient condition for this is that $\lambda_n(\mathbf{A}) \geq \lambda_1(\mathbf{B})$.

## 5.16 Projection Matrix

A square matrix $\mathbf{P}$ is a projection matrix that projects onto a vector space $\mathcal{S}$ iff

$$\mathbf{P} \text{ is idempotent} \tag{208}$$
$$\mathbf{P}\mathbf{x} \in \mathcal{S} \quad \forall\mathbf{x} \tag{209}$$
$$\mathbf{P}\mathbf{z} = \mathbf{z} \quad \forall\mathbf{z} \in \mathcal{S} \tag{210}$$

## 5.17 Single-Entry Matrix

$$\mathbf{J}^{2,3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{211}$$

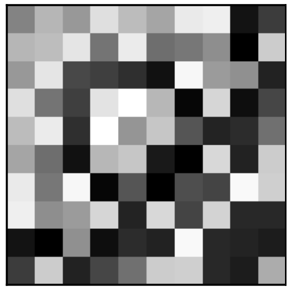
The single-entry matrix $\mathbf{J}^{iJ} \in \mathbb{R}^{n,n}$ is defined as the matrix which is zero everywhere except for the entry $(i, j)$, which is 1.

## 5.18 Singular Matrix

A square matrix that is not invertible.

$\mathbf{A} \in \mathbb{R}^{n,n}$ is singular iff $\det\mathbf{A} = 0$ iff $\mathcal{N}(A) \neq \{0\}$.

## 5.19 Symmetric Matrix



$\mathbf{A} \in \mathbb{S}^n$ is a symmetric matrix if $\mathbf{A} = \mathbf{A}^T$ (entries above diagonal are equal to entries below diagonal).

$$\begin{bmatrix} a & b & c & d & e & f \\ b & g & l & m & o & p \\ c & l & h & n & q & r \\ d & m & n & i & s & t \\ e & o & q & s & j & u \\ f & p & r & t & u & k \end{bmatrix} \tag{212}$$

### Special Properties

$$\mathbf{A} = \mathbf{A}^T \tag{213}$$

$$\mathrm{eig}(A) \in \mathbb{R}^n \tag{214}$$

$$\text{Number of “free entries”} = \frac{n(n+1)}{2} \tag{215}$$

If $\mathbf{A}$ is real, it can be decomposed into $\mathbf{A} = \mathbf{Q}^T\mathbf{D}\mathbf{Q}$ where $\mathbf{Q}$ is a real orthogonal matrix (the columns of which are eigenvectors of $\mathbf{A}$) and $\mathbf{D}$ is real and diagonal containing the eigenvalues of $\mathbf{A}$.

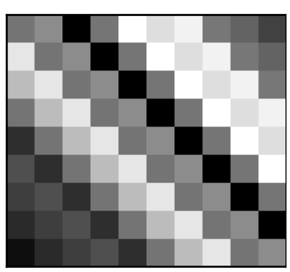
For a real, symmetric matrix with non-negative eignevalues, the eigenvalues and singular values coincide.

## 5.20 Skew-Hermitian

A matrix $\mathbf{H} \in \mathbb{C}^{m,n}$ is Skew-Hermitian iff

$$\mathbf{H} = -\mathbf{H}^H \tag{216}$$

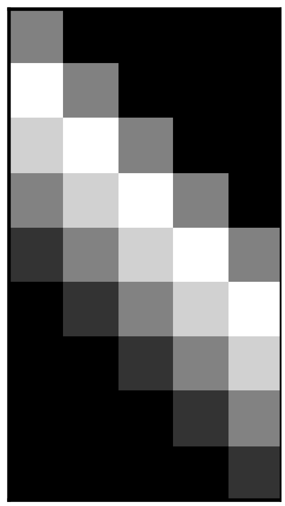
## 5.21 Toeplitz Matrix, General Form



Constant values on descending diagonals.

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & \dots & \dots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \dots & \dots & a_2 & a_1 & a_0 \end{bmatrix} \tag{217}$$

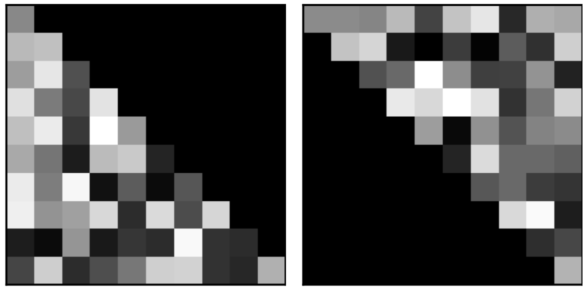
## 5.22 Toeplitz Matrix, Discrete Convolution



Constant values on main and subdiagonals.

$$\begin{bmatrix} h_m & 0 & 0 & \dots & 0 & 0 \\ \vdots & h_m & 0 & \dots & 0 & 0 \\ h_1 & \vdots & h_m & \dots & 0 & 0 \\ 0 & h_1 & \ddots & \ddots & 0 & 0 \\ 0 & 0 & h_1 & \ddots & h_m & 0 \\ 0 & 0 & 0 & \ddots & \vdots & h_m \\ 0 & 0 & 0 & \dots & h_1 & \vdots \\ 0 & 0 & 0 & \dots & 0 & h_1 \end{bmatrix} \tag{218}$$

## 5.23 Triangular Matrix



$$\begin{bmatrix} a & b & c & d & e & f \\ & g & h & i & j & k \\ & & l & m & n & o \\ & & & p & q & r \\ & & & & s & t \\ & & & & & u \end{bmatrix} \quad \begin{bmatrix} a & & & & & \\ b & g & & & & \\ c & h & l & & & \\ d & i & m & p & & \\ e & j & n & q & s & \\ f & k & o & r & t & u \end{bmatrix} \tag{219}$$

Square matrices in which all elements either above or below the main diagonal are zero. An upper (left) and a lower (right) triangular matrix are shown above.

For an upper triangular matrix $A_{ij} = 0$ whenever $i > j$; for a lower triangular matrix $A_{ij} = 0$ whenever $i < j$.

### Special Properties

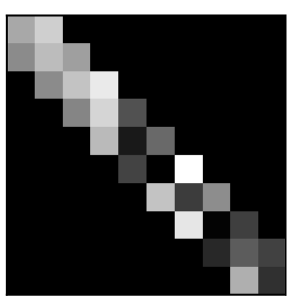
$$\mathrm{eig}(A) = \mathrm{diag}(A) \tag{220}$$

$$\det(A) = \prod_i \mathrm{diag}(A)_i \tag{221}$$

The product of two upper (lower) triangular matrices is still upper (lower) triangular.

The inverse of a nonsingular upper (lower) triangular matrix is still upper (lower) triangular.

## 5.24 Tridiagonal Matrix



$$\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & a_4 & b_4 & \ddots & \\ & & & \ddots & \ddots & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix} \tag{222}$$

A tridiagonal matrix has values on its main diagonal as well as the diagonals abutting the main, with zeros elsewhere.

A system of $n$ unknowns which can be written as

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i \tag{223}$$
$$a_1 = 0 \tag{224}$$
$$c_n = 0 \tag{225}$$

can be rewritten as

$$\begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & a_3 & b_3 & c_3 & & \\ & & a_4 & b_4 & \ddots & \\ & & & \ddots & \ddots & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{bmatrix} \tag{226}$$

This system can be solved in $O(n)$ time using the tridiagonal matrix algorithm (aka the Thomas Algorithm). The algorithm is not unconditionally stable; however, it is stable when the matrix is diagonally dominant or symmetric positive definite. A matix is diagonally dominant if for every row of the matrix the agnitude of the diagoanl entry is greater than or equal to the sum of the magnitudes of all the other non-diagonal entries in that row ($|a_{ii}| \ge \sum_{j \ne i} |a_{ij}| \; \forall i$). If uncondonitional stability is grequired, Gaussian elimination with partial pivoting is an alternative, if slower, solution method. See [9, Theorem 9.12] for full stability details.

A modified system can be solved for situations involving periodic boundary conditions, e.g.:

$$a_1 x_n + b_1 x_1 + c_1 x_2 = d_1 \tag{227}$$
$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i \quad \forall i = 2, \ldots, n-1 \tag{228}$$
$$a_n x_{n-1} + b_n x_n + c_n x_1 = d_n \tag{229}$$

Modified algorithms are also available for block tridiagonal matrices [10, §3.8]. See [11, §5.5] for a discussion of parallel solvers.

## 5.25 Unipotent

A matrix $\mathbf{A}$ is unipotent iff

$$\mathbf{A}\mathbf{A} = \mathbf{I} \tag{230}$$

### Special Properties

$$f(s\mathbf{I} + t\mathbf{A}) = \frac{1}{2}\big((\mathbf{I} + \mathbf{A})f(s+t) + (\mathbf{I} - \mathbf{A})f(s-t)\big) \tag{231}$$

## 5.26 Vandermonde Matrix

$$V = \begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ 1 & \alpha_3 & \alpha_3^2 & \dots & \alpha_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{n-1} \end{bmatrix} \tag{232}$$

Alternatively,

$$\mathbf{V}_{i,j} = \alpha_i^{j-1} \tag{233}$$

### Uses

Polynomial interpolation of data.

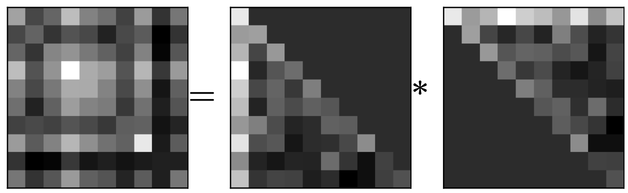
### Special Properties

$\mathbf{V}^T$ is also a Vandermone matrix.

$$\det(\mathbf{V}) = \prod_{1 \le i < j \le n} (x_j - x_i) \tag{234}$$

# 6 Matrix Decompositions

## 6.1 LLT/UTU: Cholesky Decomposition

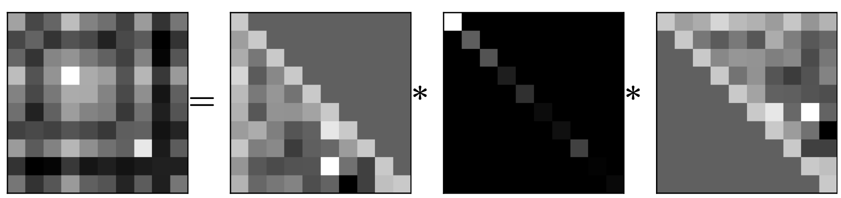


If $\mathbf{A}$ is symmetric, positive definite, square, then

$$\mathbf{A} = \mathbf{U}^T\mathbf{U} = \mathbf{L}\mathbf{L}^T \tag{235}$$

where $\mathbf{U}$ is a unique upper triangular matrix and $\mathbf{L}$ is a unique lower-triangular matrix.

## 6.2 LDL Decomposition



This is a special case of the LDM decomposition.[1] If $\mathbf{A}$ is a non-singular symmetric definite square matrix, then

$$\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T = \mathbf{L}^T\mathbf{D}\mathbf{L} \tag{236}$$

where $\mathbf{L}$ is a unit lower triangular matrix and $\mathbf{D}$ is a diagonal matrix. If $\mathbf{A} \succ 0$, then $\mathbf{D}_{ii} > 0$.

## 6.3 PCA: Principle Components Analysis

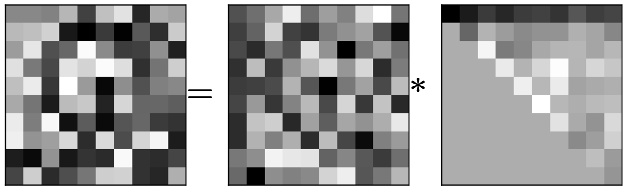
Find normalized directions in data space such that the variance of the projections of the centered data points is maximal. For centered data $\tilde{\mathbf{X}}$, the mean-square variation of data along a vector $\mathbf{x}$ is $\mathbf{x}^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\mathbf{x}$.

$$\max_{\mathbf{x}\in\mathbb{R}^n,\ \|\mathbf{x}\|_2=1} \mathbf{x}^T\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T\mathbf{x} \tag{237}$$

Taking an SVD of $\tilde{\mathbf{X}}\tilde{\mathbf{X}}^T$ gives $H = \mathbf{U}_r\mathbf{D}^2\mathbf{U}^T$, which is maximized by taking $\mathbf{x} = \mathbf{u}_1$. By repeatedly removing the first principal components and recalculating, all the principal axes can be found.

[1] TODO: Crossreference

## 6.4 QR: Orthogonal-triangular

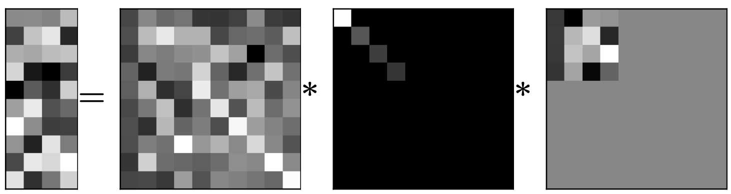


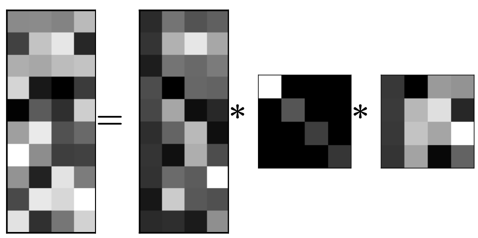
For $\mathbf{A} \in \mathbb{R}^{n,n}$, $\mathbf{A} = \mathbf{QR}$ where $\mathbf{Q}$ is orthogonal and $\mathbf{R}$ is an upper triangular matrix. If $\mathbf{A}$ is non-singular, then $\mathbf{Q}$ and $\mathbf{R}$ are uniquely defined if $\mathrm{diag}(\mathbf{R})$ are imposed to be positive.

### Algorithms

Gram-Schmidt.

## 6.5 SVD: Singular Value Decomposition





Any matrix $\mathbf{A} \in \mathbb{R}^{m,n}$ can be written as

$$\mathbf{A} = \mathbf{UDV}^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T \tag{238}$$

where

$$\mathbf{U} = \text{eigenvectors of } \mathbf{AA}^T \qquad \mathbb{R}^{m,m} \tag{239}$$

$$\mathbf{D} = \mathrm{diag}(\sigma_i) = \sqrt{\mathrm{diag}(\mathrm{eig}(\mathbf{AA}^T))} \qquad \mathbb{R}^{m,n} \tag{240}$$

$$\mathbf{V} = \text{eigenvectors of } \mathbf{A}^T\mathbf{A} \qquad \mathbb{R}^{n,n} \tag{241}$$

Let $\sigma_i$ be the non-zero singular values for $i = 1, \ldots, r$ where $r$ is the rank of $\mathbf{A}$; $\sigma_1 \geq \ldots \geq \sigma_r$.

We also have that

$$\mathbf{A}\mathbf{v}_i = \sigma_i \mathbf{u}_i \tag{242}$$

$$\mathbf{A}^T\mathbf{u}_i = \sigma_i \mathbf{v}_i \tag{243}$$

$$\mathbf{U}^T\mathbf{U} = \mathbf{I} \tag{244}$$

$$\mathbf{V}^T\mathbf{V} = \mathbf{I} \tag{245}$$

$\mathbf{D}$ can be written in an expanded form:

$$\tilde{\mathbf{D}} = \begin{bmatrix} \mathbf{D} & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{bmatrix} \tag{246}$$

The final $n-r$ columns of $\mathbf{V}$ give an orthonormal basis spanning $\mathcal{N}(\mathbf{A})$. An orthonormal basis spanning the range of $\mathbf{A}$ is given by the first $r$ columns of $\mathbf{U}$.

$$\|\mathbf{A}\|_F^2 = \text{Frobenius norm} = \text{tr}(\mathbf{A}^T\mathbf{A}) = \sum_{i=1}^r \sigma_i^2 \tag{247}$$

$$\|\mathbf{A}\|_2^2 = \sigma_1^2 \tag{248}$$

$$\|\mathbf{A}\|_* = \text{nuclear norm} = \sum_{i=1}^r \sigma_i \tag{249}$$

The **condition number** $\kappa$ of an invertible matrix $\mathbf{A} \in \mathbb{R}^{n,n}$ is the ratio of the largest and smallest singular value. Matrices with large condition numbers are closer to being singular and more sensitive to changes.

$$\kappa(\mathbf{A}) = \frac{\sigma_1}{\sigma_n} = \|A\|_2 \cdot \left\|A^{-1}\right\|_2 \tag{250}$$

## Low-Rank Approximation

Approximating $\mathbf{A} \in \mathbb{R}^{m,n}$ by a matrix $\mathbf{A}_k$ of rank $k > 0$ can be formulated as the optimization probem

$$\min_{\mathbf{A}_k \in \mathbb{R}^{m,n}} \|\mathbf{A} - \mathbf{A}_k\|_F^2 : \text{rank}\,\mathbf{A}_k = k, 1 \leq k \leq \text{rank}(\mathbf{A}) \tag{251}$$

The optimal solution of this problem is given by

$$\mathbf{A}_k = \sum_{i=1}^k \sigma_i \mathbf{u}_i \mathbf{v}_i^T \tag{252}$$

where

$$\frac{\|\mathbf{A}_k\|_F^2}{\|\mathbf{A}\|_F^2} = \frac{\sigma_1^2 + \ldots + \sigma_k^2}{\sigma_1^2 + \ldots + \sigma_r^2} \tag{253}$$

$$1 - \frac{\|\mathbf{A}_k\|_F^2}{\|\mathbf{A}\|_F^2} = \frac{\sigma_{k+1}^2 + \ldots + \sigma_r^2}{\sigma_1^2 + \ldots + \sigma_r^2} \tag{254}$$

is the fraction of the total variance in $\mathbf{A}$ explained by the approximation $\mathbf{A}_k$.

## Range and Nullspace

$$\mathcal{N}(\mathbf{A}) = \mathcal{R}(\mathbf{V}_{nr}) \tag{255}$$

$$\mathcal{N}(\mathbf{A})^\perp \equiv \mathcal{R}(\mathbf{A}^T) = \mathcal{R}(\mathbf{V}_r) \tag{256}$$

$$\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{U}_r) \tag{257}$$

$$\mathcal{R}(\mathbf{A})^\perp \equiv \mathcal{N}(\mathbf{A}^T) = \mathcal{R}(\mathbf{U}_{nr}) \tag{258}$$

where $\mathbf{V}_r$ is the first $r$ columns of $V$ and $V_n r$ are the last $[r+1, n]$ columns; similarly for $\mathbf{U}$.

### Projectors

The projection of $\mathbf{x}$ onto $\mathcal{N}(\mathbf{A})$ is $(\mathbf{V}_{nr}\mathbf{V}_{nr}^T)\mathbf{x}$. Since $\mathbf{I}_n = \mathbf{V}_r\mathbf{V}_r^T + \mathbf{V}_{nr}\mathbf{V}_{nr}^T$, $(\mathbf{I}_n - \mathbf{V}_r\mathbf{V}_r^T)\mathbf{x}$ also works. The projection of $\mathbf{x}$ onto $\mathcal{R}(\mathbf{A})$ is $(\mathbf{U}_r\mathbf{U}_r^T)\mathbf{x}$.

If $\mathbf{A} \in \mathbb{R}^{m,n}$ is full row rank ($\mathbf{A}\mathbf{A}^T \succ 0$), then the minimum distance to an affine set $\{x : \mathbf{A}\mathbf{x} = \mathbf{b}\}, \mathbf{b} \in \mathbb{R}^m$ is given by $\mathbf{x}^* = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{b}$.

If $\mathbf{A} \in \mathbb{R}^{m,n}$ is full column rank ($\mathbf{A}^T\mathbf{A} \succ 0$), then the minimum distance to an affine set $\{x : \mathbf{A}\mathbf{x} = \mathbf{b}\}, \mathbf{b} \in \mathbb{R}^m$ is given by $\mathbf{x}^* = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$.

### Computational Notes

A *numerical rank* can be estimated for the matrix as the largest $k$ such that $\sigma_k > \epsilon\sigma_1$ for $\epsilon \geq 0$.

## 6.6 Eigenvalue Decomposition for Diagonalizable Matrices

For a square, diagonalizable matrix $\mathbf{A} \in \mathbb{R}^{n,n}$

$$\mathbf{A} = U\Lambda U^{-1} \tag{259}$$

where $U \in \mathbb{C}^{n,n}$ is an invertible matrix whose columns are the eigenvectors of $\mathbf{A}$ and $\Lambda$ is a diagonal matrix containing the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{A}$ in the diagonal.

The columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ satisfy

$$\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{u}_i \quad i = 1, \ldots, n \tag{260}$$

## 6.7 Eigenvalue (Spectral) Decomposition for Symmetric Matrices

A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n,n}$ can be factored as

$$\mathbf{A} = U\Lambda U^T = \sum_i^n \lambda_i\mathbf{u}_i\mathbf{u}_i^T \tag{261}$$

where $U \in \mathbb{R}^{n,n}$ is an orthogonal matrix whose columns $\mathbf{u}_i$ are the eigenvectors of $\mathbf{A}$ and $\Lambda$ is a diagonal matrix containing the eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ of $\mathbf{A}$ in the diagonal. These eigenvalues are always real. The eigenvectors can always be chosen to be real and to form an orthonormal basis.

The columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ satisfy

$$\mathbf{A}\mathbf{u}_i = \lambda_i\mathbf{u}_i \quad i = 1, \ldots, n \tag{262}$$

## 6.8 Schur Complements

For $\mathbf{A} \in \mathbb{S}^n$, $\mathbf{B} \in \mathbb{S}^n$, $\mathbf{X} \in \mathbb{R}^{n,m}$ with $\mathbf{B} \succ 0$ and the block matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{X} \\ \mathbf{X}^T & \mathbf{B} \end{bmatrix} \tag{263}$$

and the Schur complement of $\mathbf{A}$ in $\mathbf{M}$

$$S = \mathbf{A} - \mathbf{X}\mathbf{B}^{-1}\mathbf{X}^T \tag{264}$$

Then

$$\mathbf{M} \succeq 0 \iff S \succeq 0 \tag{265}$$

$$\mathbf{M} \succ 0 \iff S \succ 0 \tag{266}$$

# 7 Eigenvalue Properties

$\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{A} \in \mathbb{R}^{n,n}$ and $u \in \mathbb{C}^n$ is a corresponding eigenvector if $\mathbf{Au} = \lambda\mathbf{u}$ and $\mathbf{u} \neq 0$. Equivalantly, $(\lambda\mathbf{I}_n - \mathbf{A})\mathbf{u} = 0$ and $\mathbf{u} \neq 0$. Eigenvalues satisfy the equation $\det(\lambda\mathbf{I}_n - \mathbf{A}) = 0$.

Any matrix $\mathbf{A} \in \mathbb{R}^{n,n}$ has $n$ eigenvalues, though some may be repeated. $\lambda_1$ is the largest eigenvalue and $\lambda_n$ the smallest.

If $\lambda$ is an eigenvalue of $\mathbf{A}$, $\lambda^2$ is an eigenvalue of $\mathbf{A}^2$.

$$\text{eig}(\mathbf{A}\mathbf{A}^T) = \text{eig}(\mathbf{A}^T\mathbf{A}) \tag{267}$$

(Note that the number of entries in $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$ may differ significantly leading to different compute times.)

$$\text{eig}(\mathbf{A}^T\mathbf{A}) \geq 0 \tag{268}$$

$$\lambda_{\min}(\mathbf{A}) \leq \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \leq \lambda_{\max}(\mathbf{A}) \quad \mathbf{x} \neq 0 \tag{269}$$

## 7.1 Weyl's Inequality

If $\mathbf{M}, \mathbf{H}, \mathbf{P} \in \mathbb{R}^{n,n}$ are Hermitian matrices and $\mathbf{M} = \mathbf{H} + \mathbf{P}$ ($\mathbf{H}$ is perturbed by $\mathbf{P}$) and $\mathbf{M}$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_n$, $\mathbf{H}$ has eigenvalues $\nu_1 \geq \cdots \geq \nu_n$, and $\mathbf{P}$ has eigenvalues $\rho_1 \geq \cdots \geq \rho_n$, then

$$\nu_i + \rho_n \leq \mu_i \leq \nu_i + \rho_1 \ \forall i \tag{270}$$

If $j + k - n \geq i \geq r + s - 1$, then

$$\nu_j + \rho_k \leq \mu_i \leq \nu_r + \rho_s \tag{271}$$

If $\mathbf{P} \succeq 0$, then $\mu_i > \nu_i \ \forall i$.

## 7.2 Estimating Eigenvalues

### 7.2.1 Gershgorin circle theorem

For $\mathbf{A} \in \mathbb{C}^{n,n}$ with entries $a_{ij}$ let $R_i = \sum_{j \neq i} |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries of the $i$-th row. Let $D(a_{ii}, R_i) \subseteq \mathbb{C}$ be a closed disc (a circle containing its boundary) centered at $a_{ii}$ with radius $R_i$. This is the Gershgorin disc.

Every eigenvalue of $\mathbf{A}$ lies within at least one of the $D(a_{ii}, R_i)$. Further, if the union of $k$ such discs is disjoint from the union of the other $n - k$ discs then the former union contains exactly $k$ and the latter $n - k$ of the eigenvalues of $\mathbf{A}$.

# 8 Norms

## 8.1 General Properties

Matrix norms satisfy some properties:

$$f(\mathbf{A}) \geq 0 \tag{272}$$

$$f(\mathbf{A}) = 0 \iff \mathbf{A} = 0 \tag{273}$$

$$f(c\mathbf{A}) = |c| f(\mathbf{A}) \tag{274}$$

$$f(\mathbf{A} + \mathbf{B}) \leq f(\mathbf{A}) + f(\mathbf{B}) \tag{275}$$

Many popular norms also satisfy "sub-multiplicativity": $f(\mathbf{AB}) \leq f(\mathbf{A}) f(\mathbf{B})$.

## 8.2 Matrices

### 8.2.1 Frobenius norm

$$\|\mathbf{A}\|_F = \sqrt{\operatorname{tr} \mathbf{A}\mathbf{A}^H} \tag{276}$$

$$= \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |\mathbf{A}_{ij}|^2} \tag{277}$$

$$= \sqrt{\sum_{i=1}^{m} \operatorname{eig}(A^H A)_i} \tag{278}$$

#### Special Properties

$$\|\mathbf{A}\mathbf{x}\|_2 \leq \|\mathbf{A}\|_F \|\mathbf{x}\|_2 \quad \mathbf{x} \in \mathbb{R}^n \tag{279}$$

$$\|\mathbf{AB}\|_F \leq \|\mathbf{A}\|_F \|\mathbf{B}\|_F \tag{280}$$

$$\left\| \mathbf{C} - \mathbf{x}\mathbf{x}^T \right\|_F^2 = \|\mathbf{C}\|_F^2 + \|\mathbf{x}\|_2^4 - 2\mathbf{x}^T \mathbf{C} \mathbf{x} \tag{281}$$

### 8.2.2 Operator Norms

For $p = 1, 2, \infty$ or other values, an operator norm indicates the maximum input-output gain of the matrix.

$$\|\mathbf{A}\|_p = \max_{\|\mathbf{u}\|_p = 1} \|\mathbf{A}\mathbf{u}\|_p \tag{282}$$

$$\|\mathbf{A}\|_1 = \max_{\|\mathbf{u}\|_1 = 1} \|\mathbf{A}\mathbf{u}\|_1 \tag{283}$$

$$= \max_{j=1,\ldots,n} \sum_{i=1}^{m} |\mathbf{A}_{ij}| \tag{284}$$

$$= \text{Largest absolute column sum} \tag{285}$$

$$\|\mathbf{A}\|_\infty = \max_{\|\mathbf{u}\|_\infty=1} \|\mathbf{A}\mathbf{u}\|_\infty \tag{286}$$
$$= \max_{j=1,\ldots,m} \sum_{i=1}^{n} |\mathbf{A}_{ij}| \tag{287}$$
$$= \text{Largest absolute row sum} \tag{288}$$

$$\|\mathbf{A}\|_2 = \text{"spectral norm"} \tag{289}$$
$$= \max_{\|\mathbf{u}\|_2=1} \|\mathbf{A}\mathbf{u}\|_2 \tag{290}$$
$$= \sqrt{\max(\text{eig}(\mathbf{A}^T\mathbf{A}))} \tag{291}$$
$$= \text{Square root of largest eigenvalue of } \mathbf{A}^T\mathbf{A} \tag{292}$$

**Special Properties**

$$\|\mathbf{A}\mathbf{u}\|_p \leq \|\mathbf{A}\|_p \|\mathbf{u}\|_p \tag{293}$$
$$\|\mathbf{A}\mathbf{B}\|_p \leq \|\mathbf{A}\|_p \|\mathbf{B}\|_p \tag{294}$$

### 8.2.3 Spectral Radius

Not a proper norm.

$$\rho(\mathbf{A}) = \text{spectral radius}(\mathbf{A}) = \max_{i=1,\ldots,n} |\,\text{eig}(\mathbf{A})_i| \tag{295}$$

**Special Properties**

$$\rho(\mathbf{A}) \leq \|\mathbf{A}\|_p \tag{296}$$
$$\rho(\mathbf{A}) \leq \min(\ \|\mathbf{A}\|_1, \|\mathbf{A}\|_\infty) \tag{297}$$

## 8.3 Vectors

$$\|\mathbf{x}\|_1 = \sum_i |\mathbf{x}_i| \qquad \text{L1-norm} \tag{298}$$
$$\|\mathbf{x}\|_p = (\sum_i |\mathbf{x}_i|^p)^{1/p} \qquad \text{P-norm} \tag{299}$$
$$\|\mathbf{x}\|_\infty = \max_i |\mathbf{x}_i| \qquad \text{L}\infty\text{-norm, L-infinity norm} \tag{300}$$

### 8.3.1 Identities

$$2\|\mathbf{u}\|_2^2 + 2\|\mathbf{v}\|_2^2 = \|\mathbf{u}+\mathbf{v}\|_2^2 + \|\mathbf{u}-\mathbf{v}\|_2^2 \qquad \text{Polarization Identity} \tag{301}$$
$$<\mathbf{x},\mathbf{y}> = \frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|_2^2 - \|\mathbf{x}-\mathbf{y}\|_2^2\right) \quad \forall \mathbf{x},\mathbf{y} \in \mathcal{V} \qquad \text{Polarization Identity} \tag{302}$$
$$\|u\|_2^2 + \|v\|_2^2 = \left\|\begin{bmatrix} u \\ v \end{bmatrix}\right\|_2^2 \tag{303}$$

### 8.3.2 Bounds

$$|\mathbf{x}^T\mathbf{y}| \leq \|\mathbf{x}\|_2\|\mathbf{y}\|_2 \qquad \text{Cauchy-Schwartz Inequality} \tag{304}$$

$$|\mathbf{x}^T\mathbf{y}| \leq \sum_{k=1}^{n} |\mathbf{x}_k\mathbf{y}_k| \leq \|\mathbf{x}\|_p\|\mathbf{x}\|_q \quad \forall p,q \geq 1 : 1/p + 1/q = 1 \qquad \text{Hölder Inequality} \tag{305}$$

For $\mathbf{x} \in \mathbb{R}^n$

$$\frac{1}{\sqrt{n}}\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1 \leq \sqrt{\mathrm{card}(\mathbf{x})}\|\mathbf{x}\|_2 \leq \sqrt{n}\|\mathbf{x}\|_2 \leq n\|\mathbf{x}\|_\infty \tag{306}$$

For any $0 < p < q$ we have that $\|\mathbf{x}\|_q \leq \|\mathbf{x}\|_p$.

# 9 | Bounds

## 9.1 Matrix Gain

$$\lambda_{\min}(\mathbf{A}^T\mathbf{A}) \leq \frac{\|\mathbf{Ax}\|_2^2}{\|\mathbf{x}\|_2^2} \leq \lambda_{\max}(\mathbf{A}^T\mathbf{A}) \tag{307}$$

$$\max_{\mathbf{x}\neq 0} \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2} = \|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}^T\mathbf{A})} \implies \mathbf{x} = u_1 \tag{308}$$

$$\min_{\mathbf{x}\neq 0} \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2} = \sqrt{\lambda_{\min}(\mathbf{A}^T\mathbf{A})} \implies \mathbf{x} = u_n \tag{309}$$

## 9.2 Rayleigh quotients

The Rayleigh quotient of $\mathbf{A} \in \mathbb{S}^n$ is given by

$$\frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \quad \mathbf{x} \neq 0 \tag{310}$$

$$\lambda_{\min}(\mathbf{A}) \leq \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \leq \lambda_{\max}(\mathbf{A}) \quad \mathbf{x} \neq 0 \tag{311}$$

$$\lambda_{\max}(A) = \max_{\mathbf{x}:\|\mathbf{x}\|_2=1} \mathbf{x}^T\mathbf{A}\mathbf{x} = u_1 \tag{312}$$

$$\lambda_{\min}(A) = \min_{\mathbf{x}:\|\mathbf{x}\|_2=1} \mathbf{x}^T\mathbf{A}\mathbf{x} = u_n \tag{313}$$

where $u_1$ and $u_n$ are the eigenvectors associated with $\lambda_{\max}$ and $\lambda_{\min}$, respectively.

# 10 Equations

## 10.1 Linear Equations

The linear equation $\mathbf{Ax} = \mathbf{y}$ with $\mathbf{A} \in \mathbb{R}^{m,n}$ admits a solution iff $\operatorname{rank}([\mathbf{Ay}]) = \operatorname{rank}(\mathbf{A})$. If this is satisfied, the set of all solutions is an affine set $\mathcal{S} = \{\mathbf{x} = \bar{\mathbf{x}} + z : z \in \mathcal{N}(\mathbf{A})\}$ where $\bar{\mathbf{x}}$ is any vector such that $\mathbf{A}\bar{\mathbf{x}} = \mathbf{y}$. The solution is unique if $\mathcal{N}(\mathbf{A}) = \{0\}$.

$\mathbf{Ax} = \mathbf{y}$ is *overdetermined* if it is tall/skinny ($m > n$); that is, if there are more equations than unknowns. If $\operatorname{rank}(\mathbf{A}) = n$ then $\dim \mathcal{N}(\mathbf{A}) = 0$, so there is either no solution or one solution. Overdetermined systems often have no solution ($\mathbf{y} \notin \mathcal{R}(\mathbf{A})$), so an approximate solution is necessary. See § 10.2.

$\mathbf{Ax} = \mathbf{y}$ is *underdetermined* if it is short/wide ($n > m$); that is, if has more unknowns than equations. If $\operatorname{rank}(\mathbf{A}) = m$ then $\mathcal{R}(\mathbf{A}) = \mathbb{R}^m$, so $\dim \mathcal{N}(\mathbf{A}) = n - m > 0$, so the set of solutions is infinite. Therefore, finding a single solution that optimizes some quantity is of interest.

$\mathbf{Ax} = \mathbf{y}$ is *square* if $n = m$. If $\mathbf{A}$ is invertible, then the equations have the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$. See § 10.3.

## 10.2 Least-Squares

For an overdetermined system we wish to find:

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{y}\|_2^2 \tag{314}$$

Since $\mathbf{Ax} \in \mathcal{R}(\mathbf{A})$, we need a point $\tilde{\mathbf{y}} = \mathbf{Ax}^* \in \mathcal{R}(\mathbf{A})$ closest to $\mathbf{y}$. This point lies in the nullspace of $\mathbf{A}^T$, so we have $\mathbf{A}^T(\mathbf{y} - \mathbf{Ax}^*) = 0$. There is always a solution to this problem and, if $\operatorname{rank}(\mathbf{A}) = n$, it is unique [12, p. 161]

$$\mathbf{x}^* = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{y} \tag{315}$$

### 10.2.1 Regularized least-squares with low-rank data

For $\mathbf{A} \in \mathbb{R}^{m,n}$, $\mathbf{y} \in \mathbb{R}^m$, $\lambda \geq 0$, the regularized least-squares problem

$$\operatorname{argmin}_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{y}\|_2^2 + \lambda\|\mathbf{x}\|_2^2 \tag{316}$$

has a closed form solution

$$\mathbf{x} = (\mathbf{A}^T\mathbf{A} + \lambda\mathbf{I})^{-1}\mathbf{A}^T\mathbf{y} \tag{317}$$

However, if $\mathbf{A}$ has a rank $r \ll \min(n, m)$ and a known low-rank decomposition $\mathbf{A} = \mathbf{LR}^T$ with $\mathbf{L} \in \mathbb{R}^{m,r}$ and $\mathbf{R} \in \mathbb{R}^{n,r}$, then we can rewrite Equation 317 as

$$\mathbf{x} = (\mathbf{R}^T\mathbf{RL}^T\mathbf{L} + \lambda\mathbf{I})^{-1}\mathbf{L}^T\mathbf{y} \tag{318}$$

This decreases the time complexity from $O(mn^2 + n^\omega)$ to $O(nr^2 + mr^2)$.

## 10.3 Minimum Norm Solutions

For undertermined systems in which $\mathbf{A} \in \mathbb{R}^{m,n}$ with $m < n$. We wish to find

$$\min_{\mathbf{x}:\mathbf{A}\mathbf{x}=\mathbf{y}} \|\mathbf{x}\|_2 \tag{319}$$

The solution $\mathbf{x}^*$ must be orthogonal to $\mathcal{N}(\mathbf{A})$, so $\mathbf{x}^* \in \mathcal{R}(\mathbf{A}^T)$, so $\mathbf{x}^* = \mathbf{A}^T c$ for some $c$. Substituting into $\mathbf{A}\mathbf{x} = \mathbf{y}$ gives $\mathbf{A}\mathbf{A}^T c = \mathbf{y}$, therefore [12, p. 162]:

$$\mathbf{x}^* = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{y} \tag{320}$$

## 10.4 The Sylvester Equation: $\mathbf{AX} + \mathbf{X}^T\mathbf{B} = \mathbf{C}$

The equation

$$\mathbf{AX} + \mathbf{X}^T\mathbf{B} = \mathbf{C} \tag{321}$$

is called a T-Sylvester equation, or *-Sylvester equation in the complex case. It can be solved using methods from, e.g.: De Terán and Dopico [13], De Terán et al. [14], Dopico et al. [15].

# 11 | Updates

## 11.1 Woodbury Identity (rank-$k$ update to inverse)

The inverse of a rank-$k$ update of some matrix $\mathbf{A}$ can be computed by doing a rank-$k$ update of $\mathbf{A}^{-1}$.

$$(\mathbf{A}+\mathbf{UCV})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{B}^{-1}+\mathbf{VA}^{-1}\mathbf{U})^{-1}\mathbf{VA}^{-1} \tag{322}$$
$$\tag{323}$$

where $\mathbf{A} \in \mathbb{R}^{n,n}$, $\mathbf{C} \in \mathbb{R}^{k,k}$, $\mathbf{U} \in \mathbb{R}^{n,k}$, $\mathbf{V} \in \mathbb{R}^{k,n}$, and $\mathbf{A}$ and $\mathbf{C}$ non-singular.

If $\mathbf{U}$ and $\mathbf{V}$ are vectors, then the Woodbury Identity reduces to the Sherman–Morrison formula (§ 11.2).

If $\mathbf{P}, \mathbf{R}$ are positive definite and $\mathbf{P} \in \mathbb{R}^{n,n}$, $\mathbf{R} \in \mathbb{R}^{k,k}$, and $\mathbf{B} \in \mathbb{R}^{k,n}$, then

$$(\mathbf{P}^{-1}+\mathbf{B}^T\mathbf{R}^{-1}\mathbf{B})^{-1} = \mathbf{P} - \mathbf{PB}^T(\mathbf{BPB}^T+\mathbf{R})^{-1}\mathbf{BP} \tag{324}$$ [16]

$$(\mathbf{P}^{-1}+\mathbf{B}^T\mathbf{R}^{-1}\mathbf{B})^{-1}\mathbf{B}^T\mathbf{R}^{-1} = \mathbf{PB}^T(\mathbf{BPB}^T+\mathbf{R})^{-1} \tag{325}$$ [16]

## 11.2 Sherman–Morrison Formula (rank-1 update to inverse)

The inverse of a rank-1 update of some matrix $\mathbf{A}$ can be computed by doing a rank-1 update of $\mathbf{A}^{-1}$.

$$(\mathbf{A}+\mathbf{uv}^T)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{uv}^T\mathbf{A}^{-1}}{1+\mathbf{v}^T\mathbf{A}^{-1}\mathbf{u}} \tag{326}$$

This is a special case of the Woodbury Identity (§ 11.1).

## 11.3 Removing a row from $\mathbf{A}^T\mathbf{A}$ ($\mathbf{A}^T\mathbf{A} \to \mathbf{A}_{\setminus i}^T\mathbf{A}_{\setminus i}$)

**Plain English:** Matrix times its transpose after eliminating row $i$ from the matrix

**Inputs:** $\mathbf{A} \in \mathbb{R}^{k,m}, \mathbf{u} \in \mathbb{R}^m, \mathbf{v} \in \mathbb{R}^n$ and $i$, the row to remove from $\mathbf{A}$

**Reduces to:** $\mathbf{C} \in \mathbb{R}^{k,l}$

**Algorithm:**

$$\mathbf{A}_{\setminus i}^T\mathbf{A}_{\setminus i} = \mathbf{A}^T\mathbf{A} - \mathbf{A}_{*i}\mathbf{A}_{*i}^T \tag{327}$$

Similarly:

$$\mathbf{A}_{\setminus i}^T y_{\setminus i} = \mathbf{A}^T y - \mathbf{A}_{*i} y_i^T \tag{328}$$

## 11.4 $\mathbf{1}_r^T \mathbf{A} \mathbf{1}_c$

**Plain English:** The sum of the elements of the matrix.

**Reduces to:** Scalar

**Notation:** For $\mathbf{A} \in \mathbb{R}^{r \times c}$, $\mathbf{1}_r$ is in $\mathbb{R}^{r \times 1}$ and $\mathbf{1}_c$ is in $\mathbb{R}^{c \times 1}$.

**Algorithm:** Traverse all the element of the matrix in order keeping track of the sum. For applications where accuracy is important and the matrices have a large dynamic range, Kahan summation or a similar technique should be used.

**Update Algorithm:** If an entry changes, subtract its old value from the sum and add its new value to the sum.

## 11.5 $\mathbf{e}_i \mathbf{A} \mathbf{e}_j$

**Plain English:** Extract element $\mathbf{A}_{ij}$ from the matrix

**Reduces to:** Scalar

**Notation:** TODO

**Algorithm:** TODO

**Update Algorithm:** TODO

## 11.6 $\mathbf{x}^T \mathbf{A} \mathbf{x}$

**Plain English:** TODO

**Reduces to**: Scalar

**Notation:** $\mathbf{A}$ must be in $\mathbb{R}^{i \times i}$. $\mathbf{x}$ is in $\mathbb{R}^{i \times 1}$.

**Algorithm:** TODO

**Update Algorithm:** We make use of the identity $(\mathbf{x}^T \mathbf{A} \mathbf{x}) = \sum_{i,j} \left( (\mathbf{x}\mathbf{x}^T) \circ \mathbf{A} \right)$. If an entry $\mathbf{A}_{i,j}$ in the matrix changes subtract its old value $\mathbf{x}_i \mathbf{x}_j \mathbf{A}_{ij}$ and add the new value $\mathbf{x}_i \mathbf{x}_j \mathbf{A}'_{ij}$. If an entry $\mathbf{x}_i$ changes TODO.

# 12 | Optimization

## 12.1 Standard Forms

**Least Squares**

$$\min_{\mathbf{x}\in\mathbb{R}^n} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \tag{329}$$

**LASSO**

$$\min_{\mathbf{b}\in\mathbb{R}^n} \left( \frac{1}{N}\|\mathbf{y} - \mathbf{X}\mathbf{b}\|_2^2 + \lambda\|\mathbf{b}\|_1 \right) \tag{330}$$

**LP: Linear program**

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^T\mathbf{x} && (331a)\\ \text{subject to} \quad & \mathbf{A}_{\text{eq}}\mathbf{x} = \mathbf{b}_{\text{eq}}, && (331b)\\ & \mathbf{A}\mathbf{x} \leq \mathbf{b} && (331c) \end{aligned}$$

**Linear Fractional Program**

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \frac{\mathbf{c}^T\mathbf{x} + a}{\mathbf{d}^T\mathbf{x} + b} && (332a)\\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b} && (332b) \end{aligned}$$

Additional constraints must ensure $\mathbf{d}^T\mathbf{x} + b$ has the same sign throughout the entire feasible region.

**QCQP: Quadratic Constrainted Quadratic Programs**

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{x}^T\mathbf{H}_0\mathbf{x} + 2\mathbf{c}_0^T\mathbf{x} + \mathbf{d}_0 && (333a)\\ \text{subject to} \quad & \mathbf{x}^T\mathbf{H}_i\mathbf{x} + 2\mathbf{c}_i^T\mathbf{x} + \mathbf{d}_i \leq 0 \quad i \in \mathcal{I}, && (333b)\\ & \mathbf{x}^T\mathbf{H}_j\mathbf{x} + 2\mathbf{c}_j^T\mathbf{x} + \mathbf{d}_j = 0 \quad j \in \mathcal{E} && (333c) \end{aligned}$$

If $\mathbf{H}_i \succeq 0 \ \forall i$, then the program is convex. In general, QCQPs are NP-Hard.

**QP: Quadratic Program**

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \frac{1}{2}\mathbf{x}^T\mathbf{H}_0\mathbf{x} + \mathbf{c}_0^T\mathbf{x} && (334a)\\ \text{subject to} \quad & \mathbf{A}_{\text{eq}}\mathbf{x} = \mathbf{b}_{\text{eq}}, && (334b)\\ & \mathbf{A}\mathbf{x} \leq \mathbf{b} && (334c) \end{aligned}$$

If $\mathbf{H}_0 \succ 0$, then the program is convex.

If only equality constraints are present, then the solution is the linear system:

$$\begin{bmatrix} \mathbf{H}_0 & \mathbf{A}^T \\ \mathbf{A} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \lambda \end{bmatrix} = \begin{bmatrix} -\mathbf{c}_0 \\ \mathbf{b} \end{bmatrix} \tag{335}$$

where $\lambda$ is a set of Lagrange multipliers.

For $\mathbf{H}_0 \succ 0$, the ellipsoid method solves the problem in polynomial time. [17] If, $\mathbf{H}_0$ is indefinite, then the problem is NP-hard [18], even if $\mathbf{H}_0$ has only one negative eigenvalue [19].

**SOCP: Second Order Cone Program (Standard Form)**

$$\min_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x} \tag{336}$$

$$\text{s.t. } \|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\|_2 \leq \mathbf{c}_i^T\mathbf{x} + \mathbf{d}_i, \quad i = 1, \ldots, m \tag{337}$$

**SOCP: Second Order Cone Program (Conic Standard Form)**

$$\min_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x} \tag{338}$$

$$\text{s.t. } (\mathbf{A}_i\mathbf{x} + \mathbf{b}_i, \mathbf{c}_i^T\mathbf{x} + \mathbf{d}_i) \in \mathcal{K}_{m_i} \quad i = 1, \ldots, m \tag{339}$$

## 12.2 Transformations

### 12.2.1 Linear-Fractional to Linear

We transform a Linear-Fractional Program

$$\underset{\mathbf{x}}{\text{maximize}} \quad \frac{\mathbf{c}^T\mathbf{x} + a}{\mathbf{d}^T\mathbf{x} + b} \tag{340a}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{340b}$$

where $\mathbf{d}^T\mathbf{x} + b$ has the same sign throughout the entire feasible region to a linear program using the Charnes–Cooper transformation [20] by defining

$$\mathbf{y} = \frac{1}{\mathbf{d}^T\mathbf{x} + b} \cdot \mathbf{x} \tag{341}$$

$$t = \frac{1}{\mathbf{d}^T\mathbf{x} + b} \tag{342}$$

to form the equivalent program

$$\underset{\mathbf{y}, t}{\text{maximize}} \quad \mathbf{c}^T\mathbf{y} + at \tag{343a}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{y} \leq \mathbf{b}t, \tag{343b}$$

$$\mathbf{d}^T\mathbf{y} + bt = 1, \tag{343c}$$

$$t \geq 0 \tag{343d}$$

We then have $\mathbf{x}^* = \frac{1}{t}\mathbf{y}$.

### 12.2.2 LP as SOCP

The linear program

$$\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{c}^T\mathbf{x} \tag{344a}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b} \tag{344b}$$

becomes can be cast as an SOCP:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{c}^T\mathbf{x} \tag{345a}$$

$$\text{subject to} \quad \|\mathbf{C}_i\mathbf{x} + \mathbf{d}_i\|_2 \leq \mathbf{b}_i - \mathbf{a}_i^T\mathbf{x} \forall i \tag{345b}$$

where $\mathbf{C}_i = 0, d_i = 0 \ \forall i$.

### 12.2.3 QCQP as SOCP

The quadratic constrainted quadratic program

$$\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{x}^T\mathbf{Q}_0\mathbf{x} + \mathbf{a}_0^T\mathbf{x} \tag{346a}$$

$$\text{subject to} \quad \mathbf{x}^T\mathbf{Q}_i\mathbf{x} + \mathbf{a}_i^T\mathbf{x} \le b_i \quad i = 1, \dots, m \tag{346b}$$

with $\mathbf{Q}_i = \mathbf{Q}_i^T \succeq 0$, $i = 0, \dots, m$ can be cast as an SOCP:

$$\underset{\mathbf{x}, t}{\text{minimize}} \quad \mathbf{a}_0^T\mathbf{x} + t \tag{347a}$$

$$\text{subject to} \quad \left\| \begin{bmatrix} 2\mathbf{Q}_0^{1/2}\mathbf{x} \\ t-1 \end{bmatrix} \right\|_2 \le t+1, \tag{347b}$$

$$\left\| \begin{bmatrix} 2\mathbf{Q}_i^{1/2}\mathbf{x} \\ b_i - \mathbf{a}_i^T\mathbf{x} - 1 \end{bmatrix} \right\|_2 \le b_i - \mathbf{a}_i^T\mathbf{x} + 1 \quad i = 1, \dots, m \tag{347c}$$

### 12.2.4 QP as SOCP

The quadratic program

$$\underset{\mathbf{x}}{\text{minimize}} \quad \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{c}^T\mathbf{x} \tag{348a}$$

$$\text{subject to} \quad \mathbf{a}_i^T\mathbf{x} \le \mathbf{b}_i \tag{348b}$$

with $\mathbf{Q} = \mathbf{Q}^T \succeq 0$ can be cast as an SOCP:

$$\underset{\mathbf{x}, \mathbf{y}}{\text{minimize}} \quad \mathbf{c}^T\mathbf{x} + y \tag{349a}$$

$$\text{subject to} \quad \left\| \begin{bmatrix} 2\mathbf{Q}^{1/2}\mathbf{x} \\ y-1 \end{bmatrix} \right\|_2 \le y+1, \tag{349b}$$

$$\mathbf{a}_i^T\mathbf{x} \le \mathbf{b}_i \quad \forall i \tag{349c}$$

### 12.2.5 Sum of L2 Norms to SOCP

$$\underset{\mathbf{x}}{\text{minimize}} \quad \sum_{i=1}^{p} \|\mathbf{A}_i\mathbf{x} - \mathbf{b}_i\|_2 \tag{350a}$$

becomes

$$\underset{\mathbf{x}, y}{\text{minimize}} \quad \sum_{i=1}^{p} y_i \tag{351a}$$

$$\text{subject to} \quad \|\mathbf{A}_i\mathbf{x} - \mathbf{b}_i\|_2 \le y_i \quad i = 1, \dots, p \tag{351b}$$

### 12.2.6 Minimax of L2 Norms to SOCP

$$\underset{\mathbf{x}}{\text{minimize}} \quad \max_{i=1,\dots,p} \|\mathbf{A}_i\mathbf{x} - \mathbf{b}_i\|_2 \tag{352a}$$

becomes

$$\underset{\mathbf{x}, y}{\text{minimize}} \quad y \tag{353a}$$

$$\text{subject to} \quad \|\mathbf{A}_i\mathbf{x} - \mathbf{b}_i\|_2 \le y \quad i = 1, \dots, p \tag{353b}$$

### 12.2.7 Hyperbolic Constraints to SOCP

For scalar $w$, a constraint of the form

$$w^2 \leq xy, \quad x \geq 0, \quad y \geq 0 \tag{354}$$

can be transformed into the SOCP constraint

[21]
$$\left\| \begin{bmatrix} 2w \\ x - y \end{bmatrix} \right\|_2 \leq x + y \tag{355}$$

For vector $\mathbf{w}$, a constraint of the form

$$\mathbf{w}^T \mathbf{w} = \|\mathbf{w}\|_2^2 \leq xy, \quad x \geq 0, \quad y \geq 0 \tag{356}$$

can be transformed into the SOCP constraint

[21, 22]
$$\left\| \begin{bmatrix} 2\mathbf{w} \\ x - y \end{bmatrix} \right\|_2 \leq x + y \tag{357}$$

Note that this implies that

$$x^{-1} \leq y \iff \left\| \begin{bmatrix} 2 \\ x - y \end{bmatrix} \right\|_2 \leq x + y \tag{358}$$

### 12.2.8 Matrix Fractional to SOCP

The problem

$$\underset{\mathbf{x}}{\text{minimize}} \quad (\mathbf{F}\mathbf{x} + \mathbf{g})^T (\mathbf{P}_0 + \mathbf{x}_1 \mathbf{P} + \ldots + \mathbf{x}_p \mathbf{P}_P)^{-1} (\mathbf{F}\mathbf{x} + \mathbf{g}) \tag{359a}$$
$$\text{subject to} \quad \mathbf{P}_0 + \mathbf{x}_1 \mathbf{P} + \ldots + \mathbf{x}_p \mathbf{P}_P > 0, \tag{359b}$$
$$\mathbf{x} \geq 0 \tag{359c}$$

where $\mathbf{P}_i = \mathbf{P}_i^T \in \mathbb{R}^{n,n}$, $\mathbf{F} \in \mathbb{R}^{n,p}$, $\mathbf{g} \in \mathbb{R}^n$, and $\mathbf{x} \in \mathbb{R}^p$ can be transformed into the SOCP where $t_i \in \mathbb{R}, \mathbf{y}_i \in \mathbb{R}^n$:

$$\underset{\mathbf{x}, t}{\text{minimize}} \quad t_0 + \ldots + t_p \tag{360a}$$

[21]
$$\text{subject to} \quad \mathbf{P}_0^{1/2} \mathbf{y}_0 + \ldots + \mathbf{P}_p^{1/2} \mathbf{y}_p = \mathbf{F}\mathbf{x} + \mathbf{g}, \tag{360b}$$
$$\left\| \begin{bmatrix} 2\mathbf{y}_0 \\ t_0 - 1 \end{bmatrix} \right\|_2 \leq t_0 + 1, \tag{360c}$$
$$\left\| \begin{bmatrix} 2\mathbf{y}_i \\ t_i - x_i \end{bmatrix} \right\|_2 \leq t_i + x_i \quad i = 1, \ldots, p \tag{360d}$$

### 12.2.9 Fractional Objective to SOCP

Convert

$$\underset{\mathbf{x}}{\text{minimize}} \quad \frac{f(x)^2}{g(x)} \tag{361a}$$
$$\text{subject to} \quad g(x) > 0 \tag{361b}$$

to

$$\begin{aligned}
\underset{\mathbf{x},t}{\text{minimize}} \quad & t && (362a)\\
\text{subject to} \quad & f(x)^2 \leq tg(y), && (362b)\\
& g(y) > 0, && (362c)\\
& t \geq 0 && (362d)
\end{aligned}$$

and apply Equation 357.

### 12.2.10 Chance-Constrained LP to SOCP

The problem

$$\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^T\mathbf{x} && (363a)\\
\text{subject to} \quad & \text{Prob}\{\mathbf{a}_i^T\mathbf{x} \leq \mathbf{b}_i\} \geq p_i \quad i = 1, \dots, m && (363b)
\end{aligned}$$

where $p_i > 0.5$ and all $\mathbf{a}_i$ are independent normal random vectors with expected values $\bar{\mathbf{a}}_i$ and covariance matrices $\Sigma_i \succ 0$, can be transformed into the SOCP:

$$\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^T\mathbf{x} && (364a)\\
\text{subject to} \quad & \bar{\mathbf{a}}_i^T\mathbf{x} \leq b_i - \Phi^{-1}(p_i)\left\|\Sigma_i^{1/2}\mathbf{x}\right\|_2 \quad i = 1, \dots, m && (364b)
\end{aligned}$$

where $\Phi^{-1}(p)$ is the inverse cumulative probability distribution of a standard normal variable.

Likewise, the problem

$$\begin{aligned}
\underset{\mathbf{x}}{\text{maximize}} \quad & \mathbf{c}^T\mathbf{x} && (365a)\\
\text{subject to} \quad & \text{Prob}\{\mathbf{a}_i^T\mathbf{x} \leq \mathbf{b}_i\} \leq p_i \quad i = 1, \dots, m && (365b)
\end{aligned}$$

transforms to

$$\begin{aligned}
\underset{\mathbf{x}}{\text{maximize}} \quad & \mathbf{c}^T\mathbf{x} && (366a)\\
\text{subject to} \quad & \bar{\mathbf{a}}_i^T\mathbf{x} \geq \Phi^{-1}(1 - p_i)\left\|\Sigma_i^{1/2}\mathbf{x}\right\|_2 \quad i = 1, \dots, m && (366b)
\end{aligned}$$

provided $p_i \leq 0.5$.

### 12.2.11 Robust LP with Box Uncertainty as LP

The problem

$$\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^T\mathbf{x} && (367a)\\
\text{subject to} \quad & \mathbf{a}_i^T\mathbf{x} \leq b_i \quad \forall \mathbf{a}_i \in \{\hat{\mathbf{a}}_i + \rho_i\mathbf{u} : \|\mathbf{u}\|_\infty \leq 1\} \quad i = 1, \dots, m && (367b)
\end{aligned}$$

is equivalent to

$$\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^T\mathbf{x} && (368a)\\
\text{subject to} \quad & \hat{\mathbf{a}}_i^T\mathbf{x} + \rho_i\|\mathbf{x}\|_1 \leq b_i \quad i = 1, \dots, m && (368b)
\end{aligned}$$

which is equivalent to:

$$\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & \mathbf{c}^T\mathbf{x} && (369a)\\
\text{subject to} \quad & \hat{\mathbf{a}}_i^T\mathbf{x} + \rho_i\sum_{j=1}^{n}\mathbf{u}_j \leq b_i \qquad i = 1, \dots, m, && (369b)\\
& -\mathbf{u}_j \leq \mathbf{x}_j \leq \mathbf{u}_j \quad j = 1, \dots, n && (369c)
\end{aligned}$$

### 12.2.12 Robust LP with Ellipsoidal Uncertainty as SOCP

The problem

$$\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{c}^T\mathbf{x} \tag{370a}$$
$$\text{subject to} \quad \mathbf{a}_i^T\mathbf{x} \leq b_i \quad \forall \mathbf{a}_i \in \{\hat{\mathbf{a}}_i + \mathbf{R}_i\mathbf{u} : \|\mathbf{u}\|_2 \leq 1\} \quad i = 1, \ldots, m \tag{370b}$$

is equivalent to

$$\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{c}^T\mathbf{x} \tag{371a}$$
$$\text{subject to} \quad \hat{\mathbf{a}}_i^T\mathbf{x} + \left\|\mathbf{R}_i^T\mathbf{x}\right\|_2 \leq b_i \quad i = 1, \ldots, m \tag{371b}$$

### 12.2.13 Square Root as SOCP

$$\sqrt{x} \geq t \iff x \geq t^2 \iff \left\|\begin{matrix} 1-x \\ 2t \end{matrix}\right\|_2 \leq 1 + x \tag{372}$$

The problem

$$\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{c}^T\mathbf{x} \tag{373a}$$
$$\text{subject to} \quad \mathbf{a}_i^T\mathbf{x} \leq b_i \quad \forall \mathbf{a}_i \in \{\hat{\mathbf{a}}_i + \mathbf{R}_i\mathbf{u} : \|\mathbf{u}\|_2 \leq 1\} \quad i = 1, \ldots, m \tag{373b}$$

is equivalent to

$$\underset{\mathbf{x}}{\text{minimize}} \quad \mathbf{c}^T\mathbf{x} \tag{374a}$$
$$\text{subject to} \quad \hat{\mathbf{a}}_i^T\mathbf{x} + \left\|\mathbf{R}_i^T\mathbf{x}\right\|_2 \leq b_i \quad i = 1, \ldots, m \tag{374b}$$

## 12.3 Useful Problems

$$\text{average}(\mathbf{v}) = \min_{x\in\mathbb{R}} \|\mathbf{v} - x\mathbf{1}\|_2^2 \tag{375}$$
$$\text{median}(\mathbf{v}) = \min_{x\in\mathbb{R}} \|\mathbf{v} - x\mathbf{1}\|_1 \tag{376}$$

# 13 Algorithmics

## 13.1 Time Complexities

| Operation | Input | Output | Algorithm | Time |
|---|---|---|---|---|
| Matmult | $\mathbf{A}, \mathbf{B} \in n \times n$ | $n \times n$ | Schoolbook | $O(n^3)$ |
| | | | Strassen [23] | $O(n^{2.807})$ |
| | | | Best | $O(n^\omega)$ |
| Matmult | $\mathbf{A} \in n \times m, \mathbf{B} \in m \times p$ | $n \times p$ | Schoolbook | $O(nmp)$ |
| Inversion | $\mathbf{A} \in n \times n$ | $n \times n$ | Gauss–Jordan elimination | $O(n^3)$ |
| | | | Strassen [23] | $O(n^{2.807})$ |
| | | | Best | $O(n^\omega)$ |
| SVD | $\mathbf{A} \in m \times n$ | $m \times m, m \times n, n \times n$<br>$m \times r, r \times r, n \times r$ | | $O(mn^2)$<br>$(m \geq n)$ |
| Determinant | $\mathbf{A} \in n \times n$ | Scalar | Laplace expansion | $O(n!)$ |
| | | | Division-free [24] | $O(n!)$ |
| | | | LU decomposition | $O(n^3)$ |
| | | | Integer preserving [25] | $O(n^3)$ |
| Back substitution | $\mathbf{A}$ triangular | $n$ solutions | Back substitution | $O(n^2)$ |

## 13.2 The $\omega$ Exponent

The lower bound on matmult time complexity is $O(n^\omega)$, where $\omega$ is an unknown constant bounded by $2 \leq \omega \leq 2.3728596$ (Table 13.1 lists the known upper bound on $\omega$ over time). Algorithms achieving lower values of $\omega$ tend to be less efficient in practice for all but the largest matrices. Of the algorithms with times of less than $O(n^3)$, only the Strassen algorithm [23] has seen serious attempts at optimized implementation. Most matmult implementations use highly optimized variants of the standard $O(n^3)$ algorithm. At this point, memory and bus speeds dominate the performance of implementations, so simple Big-O notation cannot be used to reliably compare matmult performances.

The time complexity for solving sparse linear systems was bounded by $\omega$ until recently, when randomized methods were used to obtain a bound of $O(n^{2.331645})$ [26].

| **Name** | **Year** | $\omega$ |
|---|---|---|
| Standard | - | 3 |
| Strassen [23] | 1969 | 2.807 |
| Pan [27] | 1978 | 2.796 |
| Bini et al. [28] | 1979 | 2.78 |
| Schönhage [29] | 1981 | 2.548 |
| Schönhage [29] | 1981 | 2.522 |
| Romani [30] | 1982 | 2.517 |
| Coppersmith and Winograd [31] | 1982 | 2.496 |
| Strassen [32] | 1986 | 2.479 |
| Coppersmith and Winograd [33] | 1990 | 2.376 |
| Williams [34] | 2012 | 2.37293 |
| Williams [34] | 2012 | 2.37287[1] |
| Le Gall [35] | 2014 | 2.3728642 |
| Le Gall [35] | 2014 | 2.3728640 |
| Le Gall [35] | 2014 | 2.3728639 |
| Alman and Williams [36] | 2020 | 2.3728596 |

Table 13.1: Upper bounds on the value of $\omega$ over time

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# Index